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ARTICLE

Estimation of a class of population mean estimators in stratified sampling under a linear cost function[1]

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Abstract

This paper addresses the problem of estimating the population mean of the study variable using auxiliary information on an auxiliary variable in stratified random sampling. A class of transformed ratio-product type estimators is introduced, with expressions for the biases and mean squared errors (MSEs) of the proposed class of estimators derived up to the first order of approximation. Some estimators are particular members of this introduced class, and many new estimators can be generated from the suggested family of estimators. Additionally, a linear cost function is introduced, and the MSEs and optimum values for the entire family of estimators are determined. The proposed family of estimators is theoretically compared with competing estimators, and conditions under which the suggested estimators are more efficient than others are also established. An empirical study is conducted to support the suggested family of estimators. The numerical illustration demonstrated the substantial practical utility of the theoretical findings in real-world applications. The estimator with the lowest MSE is recommended for various statistical applications and for analyzing large datasets in interdisciplinary research involving statistics and data science.

Keywords: Study Variable; Auxiliary Variable; Stratified Sampling; Cost Function; MSE; PRE.

1. Introduction

In sampling theory, it has been well established that the use of auxiliary information always provides the improvement in the precision of an estimator. The auxiliary information can be utilized at both selection stages as well as at estimation stage of sampling procedure. Ratio, product and regression methods of estimation for estimating population parameters are good examples when auxiliary information is used at the estimation stage. Here by auxiliary information, we mean the additional or supplementary information available about a variable, highly correlated with the variable of study. The ratio method of estimation often results in increased precision if there is a positive correlation between study variable Y and the auxiliary variable X and regression of Y on X is linear and passes through the origin. Regression method is used if regression line does not pass through the origin. In case of negative correlation between aforesaid variables and regression line passing through origin, product method is used. In order to improve the efficiency of the suggested estimator, researchers have occasionally employed various transformations for some already known parameters of auxiliary variable.

It is an economical approach since it made better use of auxiliary variables that were previously known without increasing the survey cost. If the population under consideration is homogeneous for the characteristic under study, the simple random sampling (SRS) is the most suitable sampling technique for estimating the parameters under investigation. It is well established in SRS that the sampling variance of sample mean is directly proportional to the population variance of the study variable. Thus, if the population is heterogeneous for the characteristic under study, the sampling variance of the sample mean estimator will be high, thereby making the estimator inefficient. Therefore, when the population is heterogeneous, stratified random sampling is used. Here we shall propose a family of estimators of population mean using transformation on study as well as auxiliary variable under stratified sampling scheme. Stratification is one of the sampling designs that are able to yield increased precision. In stratified sampling, the heterogeneous population is divided into various strata which are homogeneous itself, and samples are taken from each stratum using the method of simple random sampling. In literature, parameters such as population mean, median, standard deviation, coefficient of variation, coefficient of skewness, coefficient of kurtosis, sample size etc. of h^{th} stratum, of auxiliary variable have been utilized for improving the estimation of population mean $\overline{Y}$ of study variable in various forms.

In stratified random sampling, Hansen *et al.* (1946) and Kaur (1985) proposed ratio estimators using the population mean $\overline{X}$ of X. Diana (1993) evaluated the MSE of the estimators up to the k^{th} order of approximation and proposed a class of estimators of $\overline{Y}$ utilizing one X in the stratified random sampling. For more extensive study, one may go through the work of Upadhyaya and Singh (1999). Utilizing the different values of known parameters of X, many authors worked on the improvement of the efficiencies of the estimators of $\overline{Y}$. Kadilar and Cingi (2005) suggested some ratio-regression type estimators using parameters of X. Koyuncu and Kadilar (2009) worked on some efficient class of estimators of $\overline{Y}$ using known auxiliary parameters and shown considerable improvement over existing estimators while Singh and Vishwakarma (2010) introduced a generalized method for the efficient estimation of $\bar{Y}$ in stratified sampling using known functions of X. A generalized family of transformed ratio-product estimators in sample surveys was proposed by Upadhyaya *et al.* (2011) and in order to estimate the finite $\bar{Y}$ in stratified random sampling. For more extensive study, one may go through the work of Chaudhary (2009), Yadav (2018), Yadav and Tailor (2020), Bhushan *et al.* (2023), Chaudhary *et al.* (2023) and others. Yadav *et. al* (2024) introduce a novel estimator for stratified random sampling, improving accuracy and efficiency of population mean estimates. The estimator outperforms traditional estimators under specific conditions. The study supports the theoretical findings and suggests its practical applicability in resource allocation and cost management. The study by Zaagan *et al.* (2024) explores an efficient method to estimate the population mean in stratified random sampling. Using a linear cost function, the authors develop an approach that optimizes cost-effectiveness and precision. This method incorporates an estimator designed to minimize mean square error (MSE) and improve precision relative to other existing methods. This approach can be particularly useful across various fields, including agriculture, economics, and medical sciences, where cost constraints play a role in large-scale data collection.

Building on this work, Verma *et al.* (2025a) proposed an uncertainty-based framework that explicitly captured estimation variability, resulting in improved efficiency and lower mean squared error, as demonstrated through simulation and numerical studies. Verma *et al.* (2025b) then developed an optimal estimation approach using auxiliary attribute information in stratified random sampling, which consistently outperformed traditional estimators. To address practical survey constraints, Verma *et al.* (2025c) incorporated a linear cost function and showed that optimal stratum-wise allocation enhances efficiency under fixed budgets. Extending this research further, Verma *et al.* (2025d) examined the joint estimation of two population means and introduced optimal estimators that achieved lower mean squared error than existing methods.

In this paper, we have proposed a family of estimators considering the transformations on both Y and X under stratified random sampling to estimate $\bar{Y}$ more efficiently. The bias and MSE of the proposed family are obtained up to the first degree of approximation. The suggested family of estimators is very wide. Some well-known estimators are the members of the suggested class of estimators and several estimators can be generated from the suggested family of estimators. The study of the properties of the introduced class of estimators has been presented in different sections.

2. Review of Existing Literature, Notations and Terminologies

Consider a finite population $U = \{U_1, U_2, \dots . U_N\}$ of N units. Assume that the population of size N is divided into L strata with N_h elements in the h^{th} stratum such that $N = \sum_{h=1}^{L} N_h$. Let n_h be the size of the sample drawn from h^{th} stratum of size N_h by using simple random sampling without replacement (SRSWOR) such that sample size $n = \sum_{h=1}^{L} n_h$. Let (y, x) be the study and the auxiliary variables, respectively, assuming values y_{hi} and x_{hi} for the i^{th} unit in h^{th} stratum.

Let $W_h = \left(\frac{N_h}{N}\right)$ be the stratum weight, $f_h = \left(\frac{n_h}{N_h}\right)$ be the sampling fraction, $\left[\bar{Y}_h = (1/N_h)\sum_{i=1}^{N_h} y_{hi}, \bar{X}_h = (1/N_h)\sum_{i=1}^{N_h} x_{hi}\right]$ and $\left[\bar{y}_h = (1/n_h)\sum_{i=1}^{n_h} y_{hi}, \bar{x}_h = (1/n_h)\sum_{i=1}^{n_h} x_{hi}\right]$ be the population means and sample means of Y and X respectively. Assume that $\bar{X} = \sum_{h=1}^{L} W_h \bar{X}_h$ is known, and our purpose is to estimate the population mean $\bar{Y} = \sum_{h=1}^{L} W_h \bar{Y}_h$ of Y.

When the population mean $\bar{X}$ of X is known, the usual combined ratio, product and regression estimators for estimating the population mean $\bar{Y}$ of the study variable y are respectively defined by,

$$\bar{y}_{rc} = \bar{y}_{st}\left(\frac{\bar{X}}{\bar{x}_{st}}\right) \tag{2.1}$$

[see Hansen *et al.* (1946)]

$$\bar{y}_{pc} = \bar{y}_{st}\left(\frac{\bar{x}_{st}}{\bar{X}}\right) \tag{2.2}$$

$$\bar{y}_{lrc} = \bar{y}_{st} + b\left(\bar{X} - \bar{x}_{st}\right) \tag{2.3}$$

We know that,

$$\mathrm{Var}\left(\bar{y}_{st}\right) = \sum_{h=1}^{L} W_h^2 \gamma_h S_{yh}^2 \tag{2.4}$$

where $S_{yh}^2 = \{1/(N_h - 1)\}\sum_{i=1}^{N_h}\left(y_{hi} - \bar{Y}_h\right)^2$ is the population mean square/variance of Y.

The MSEs of the estimators $\bar{y}_{rc}$, $\bar{y}_{pc}$ *and* $\bar{y}_{lrc}$ are respectively given by

$$\mathrm{MSE}\left(\bar{y}_{rc}\right) = \sum_{h=1}^{L} W_h^2 \gamma_h \left[S_{yh}^2 + R^2 S_{xh}^2 - 2RS_{yxh}\right] \tag{2.5}$$

$$\mathrm{MSE}\left(\bar{y}_{pc}\right)=\sum_{h=1}^{L} W_h^2\gamma_h\left[S_{yh}^2+R^2S_{xh}^2+2RS_{yxh}\right] \tag{2.6}$$

$$\mathrm{Var}\left(\bar{y}_{lrc}\right)=\left(\sum_{h=1}^{L} W_h^2\gamma_h S_{yh}^2\right)-\frac{\left(\sum_{h=1}^{L} W_h^2\gamma_h S_{yxh}\right)^2}{\left(\sum_{h=1}^{L} W_h^2\gamma_h S_{xh}^2\right)} \tag{2.7}$$

where, $\gamma_h=\left(1\text{-}f_h\right)/n_h$, $R=\left(\bar{Y}/\bar{X}\right)$.

Note: Many old and new estimators and different families of estimators are the members of the introduced class of estimators, therefore, we have not presented various estimators in the review of literature.

3. The proposed family of estimators

We have suggested a class of estimators for population mean $\bar{Y}$ from which we can generate many estimators by putting the suitable values of the scalars $\left(a,c,d,\phi,\eta\right)$. The form of the suggested class of estimators is defined as

$$\bar{y}_{RP}^{(c)}=\bar{u}_{st}\left[\alpha_1\left(\frac{\bar{V}}{\bar{v}_{st}}\right)^{\phi}+\alpha_2\left(\frac{\bar{v}_{st}}{\bar{V}}\right)^{\eta}\right]-a \tag{3.1}$$

where α_1 *and* α_2 *are suitably chosen scalars* such that $\alpha_1+\alpha_2\neq 1$.

Let us consider the transformation on both study variable as well as auxiliary variable such that

$$\bar{u}_{st}=\bar{y}_{st}+a$$

and $\bar{v}_{st}=c\bar{x}_{st}+d\bar{X}$

$$E\left(\bar{u}_{st}\right)=E\left(\bar{y}_{st}+a\right)\Rightarrow\left(\bar{Y}+a\right)=\bar{U}$$

$$E\left(\bar{v}_{st}\right)=E\left(c\bar{x}_{st}+d\bar{X}\right)\Rightarrow\left(c+d\right)\bar{X}=\bar{V}$$

Let $\bar{u}_{st}=\bar{U}\left(1+e_0\right)$

$$\bar{v}_{st}=\bar{V}\left(1+e_1\right)$$

Such that

$$\left.\begin{aligned}
&E\left(e_0\right)=E\left(e_1\right)=0\\
&E\left(e_0^2\right)=E\left[\frac{\bar{u}_{st}-\bar{U}}{\bar{U}}\right]^2=\frac{1}{\bar{U}^2}Var\left(\bar{u}_{st}\right)=\frac{1}{\bar{U}^2}Var\left(\bar{y}_{st}+a\right)=\frac{1}{\bar{U}^2}Var\left(\bar{y}_{st}\right)=\frac{1}{\bar{U}^2}\sum_{h=1}^{L}W_h^2\gamma_hS_{yh}^2\\
&E\left(e_1^2\right)=E\left[\frac{\bar{v}_{st}-\bar{V}}{\bar{V}}\right]^2=\frac{1}{\bar{V}^2}Var\left(\bar{v}_{st}\right)=\frac{1}{\bar{V}^2}Var\left(c\bar{x}_{st}+d\bar{X}\right)=\frac{c^2}{\bar{V}^2}Var\left(\bar{x}_{st}\right)=\frac{c^2}{\bar{V}^2}\sum_{h=1}^{L}W_h^2\gamma_hS_{xh}^2\\
&E\left(e_0e_1\right)=E\left[\left(\frac{\bar{u}_{st}-\bar{U}}{\bar{U}}\right)\left(\frac{\bar{v}_{st}-\bar{V}}{\bar{V}}\right)\right]=\frac{1}{\bar{U}\bar{V}}Cov\left(\bar{u}_{st},\bar{v}_{st}\right)=\frac{1}{\bar{U}\bar{V}}E\left[\left(\bar{y}_{st}-\bar{Y}\right)\left(c\bar{x}_{st}-c\bar{X}\right)\right]\\
&\qquad=\frac{c}{\bar{U}\bar{V}}Cov\left(\bar{y}_{st},\bar{x}_{st}\right)=\frac{c}{\bar{U}\bar{V}}\sum_{h=1}^{L}W_h^2\gamma_hS_{yxh}
\end{aligned}\right\}\quad(3.2)$$

Expressing $\bar{y}_{RP}^{(c)}$ in terms of e's, we have

$$\bar{y}_{RP}^{(c)}=\bar{U}\left(1+e_0\right)\left[\alpha_1\left(1+e_1\right)^{-\phi}+\alpha_2\left(1+e_1\right)^{\eta}\right]-a$$

$$=\bar{U}(1+e_0)\left[\alpha_1\left(1-\phi e_1+\frac{\phi(\phi+1)}{2}e_1^2\ -\cdots\right)+\alpha_2\left(1+(e_1+\frac{\eta(\eta+1)}{2}e_1^2+\cdots\right)\right]-a$$

$$=\bar{U}\left[\left(\alpha_1+\alpha_2\right)+\left(\alpha_1+\alpha_2\right)e_0-\left(\alpha_1\phi-\alpha_2\eta\right)e_1+\left\{\alpha_1\phi\left(\phi+1\right)-\alpha_2\eta\left(\eta-1\right)\right\}\frac{e_1^2}{2}-\left(\alpha_1\phi-\alpha_2\eta\right)e_0e_1+\ldots\right]-a$$

$$\left(\bar{\mathrm{y}}_{\mathrm{RP}}^{(\mathrm{c})}-\bar{\mathrm{Y}}\right)=\left(\bar{\mathrm{Y}}+\mathrm{a}\right)-\left[\begin{aligned}&\left(\alpha_1+\alpha_2\right)+\left(\alpha_1+\alpha_2\right)e_0-\left(\alpha_1\phi-\alpha_2\eta\right)e_1+\left\{\alpha_1\phi\left(\phi+1\right)-\alpha_2\eta\left(\eta-1\right)\right\}\frac{e_1^2}{2}\\&\qquad-\left(\alpha_1\phi-\alpha_2\eta\right)e_0e_1+\ldots\end{aligned}\right]-a\quad(3.3)$$

Neglecting the terms having power greater than two and taking expectation on both sides, we get the bias of the $\bar{y}_{RP}^{(c)}$ as

$$B\left(\bar{y}_{RP}^{(c)}\right)=\left(\bar{Y}+a\right)\left[\begin{aligned}&\left(\alpha_1+\alpha_2-1\right)+\left\{\frac{\alpha_1\phi\left(\phi+1\right)-\alpha_2\eta\left(\eta-1\right)}{2}\right\}\frac{c^2}{\bar{V}^2}\sum_{h=1}^{L}W_h^2\gamma_hS_{xh}^2\\&\qquad-\left(\alpha_1\phi-\alpha_2\eta\right)\frac{c}{\bar{U}\bar{V}}\sum_{h=1}^{L}W_h^2\gamma_hS_{yxh}\end{aligned}\right]\quad(3.4)$$

Squaring both sides of (3.3) and neglecting the terms having power greater than two, we have

$$\left(\bar{y}_{RP}^{(c)}-\bar{Y}\right)^2=\left(\bar{Y}+a\right)^2\left[\begin{aligned}&\left(\alpha_1+\alpha_2\right)+\left(\alpha_1+\alpha_2\right)e_0-\left(\alpha_1\phi-\alpha_2\eta\right)e_1+\left\{\alpha_1\phi\left(\phi+1\right)-\alpha_2\eta\left(\eta-1\right)\right\}\frac{e_1^2}{2}\\&\qquad-\left(\alpha_1\phi-\alpha_2\eta\right)e_0e_1-1\end{aligned}\right]^2$$

$$\left(\overline{y}_{RP}^{(c)} - \overline{Y}\right)^2 = \left(\overline{Y}+a\right)^2 \begin{bmatrix} 1+\left(\alpha_1+\alpha_2\right)^2\left(1+e_0^2+2e_0\right)+\left(\alpha_1\phi - \alpha_2\eta\right)^2 e_1^2+\left(\alpha_1\phi - \alpha_2\eta\right)\left(2e_1+2e_0e_1\right) \\ -2\left(\alpha_1+\alpha_2\right)\left(1+e_0\right)-\left\{\alpha_1\phi\left(\phi+1\right)-\alpha_2\eta\left(\eta - 1\right)\right\}\left\{e_1^2-\left(\alpha_1+\alpha_2\right)e_1^2\right\} \\ -2\left(\alpha_1+\alpha_2\right)\left(\alpha_1\phi - \alpha_2\eta\right)\left(e_1+2e_0e_1\right) \end{bmatrix}^2 \quad (3.5)$$

Now, taking expectations on both sides on (3.5) and using the results of (3.2), we get the mean squared error (MSE) of $\overline{y}_{RP}^{(c)}$ to the first degree of approximation as

$$\mathrm{MSE}\left(\overline{\mathrm{y}}_{\mathrm{RP}}^{(\mathrm{c})}\right)=\overline{\mathrm{U}}^2\left[1+\alpha_1^2Z_1+\alpha_2^2Z_2+2\alpha_1\alpha_2Z_3-2\alpha_1Z_4-2\alpha_2Z_5\right] \quad (3.6)$$

where $A=\sum_{h=1}^{L}W_h^2\gamma_hS_{yh}^2$, $B=\sum_{h=1}^{L}W_h^2\gamma_hS_{xh}^2$, $E=\sum_{h=1}^{L}W_h^2\gamma_hS_{yxh}$,

$$\mathrm{Z}_1=1+\frac{A}{\overline{U}^2}+\left\{\phi^2+\phi\left(\phi+1\right)\right\}\frac{c^2}{\overline{V}^2}B-4\phi\frac{c}{\overline{U}\overline{V}}E$$

$$\mathrm{Z}_2=1+\frac{A}{\overline{U}^2}+\left\{\eta^2-\eta\left(\eta-1\right)\right\}\frac{c^2}{\overline{V}^2}B+4\eta\frac{c}{\overline{U}\overline{V}}E$$

$$\mathrm{Z}_3=1+\frac{A}{\overline{U}^2}-\left\{\phi\eta+\frac{\eta\left(\eta-1\right)}{2}+\frac{\phi\left(\phi+1\right)}{2}\right\}\frac{c^2}{\overline{V}^2}B+2\left(\eta-\phi\right)\frac{c}{\overline{U}\overline{V}}E \quad \mathrm{Z}_4=1+\left\{\frac{\phi\left(\phi+1\right)}{2}\right\}\frac{c^2}{\overline{V}^2}B-\phi\frac{c}{\overline{U}\overline{V}}E,$$

$$\mathrm{Z}_5=1-\left\{\frac{\eta\left(\eta-1\right)}{2}\right\}\frac{c^2}{\overline{V}^2}B-\eta\frac{c}{\overline{U}\overline{V}}E$$

3.1 Optimum choice of scalars

Minimized $\mathrm{MSE}\left(\overline{\mathrm{y}}_{\mathrm{RP}}^{(\mathrm{c})}\right)$ at (2.6) w.r.t. α_1 *and* α_2, we get

$$\alpha_1=\left(\frac{\mathrm{Z}_2\mathrm{Z}_4-\mathrm{Z}_3\mathrm{Z}_5}{\mathrm{Z}_1\mathrm{Z}_2-Z_3^2}\right)=\alpha_{1opt.}\left(\text{say}\right) \quad (3.7)$$

$$\alpha_2=\left(\frac{\mathrm{Z}_1\mathrm{Z}_5-\mathrm{Z}_3\mathrm{Z}_4}{\mathrm{Z}_1\mathrm{Z}_2-Z_3^2}\right)=\alpha_{2opt.}\left(\text{say}\right) \quad (3.8)$$

Thus, the resulting minimum MSE of $\overline{y}_{RP}^{(c)}$ is given by

$$\min\ MSE\left(\bar{y}_{RP}^{(c)}\right) = \bar{U}^2\left[1 - \frac{\left(Z_1Z_5^2 + Z_2Z_4^2 - 2Z_3Z_4Z_5\right)}{\left(Z_1Z_2 - Z_3^2\right)}\right]$$

4. Some members of the suggested class of estimators

There are many well-known estimators in literature which are members of the suggested class of estimators given in (2.1) by substituting the suitable values of the scalars $\left(a,c,d,\phi,\eta\right)$ and many new estimators can also be generated from the family of estimators. Some estimators which are the members of the proposed class of the estimators are listed in Table 1.

Table 1. Some members of the proposed family of estimators for different values of $\left(a,c,d,\alpha_1,\alpha_2,\phi,\eta\right)$

Estimators		c	d	α_1	α_2	ϕ	η
$\bar{y}_{rc} = \bar{y}_{st}\left(\frac{\bar{X}}{\bar{x}_{st}}\right)$ Usual combined Ratio Estimator		1	0	1	0	1	η
$\bar{y}_{pc} = \bar{y}_{st}\left(\frac{\bar{x}_{st}}{\bar{X}}\right)$ Usual combined Product Estimator		0	1	1	0	1	η
$T_1 = \alpha_1\bar{y}_{st}$ Searls (1964) Estimator		0	1	1	0	0	η
$T_2 = \alpha_1\bar{y}_{st}\left(\frac{\bar{x}_{st}}{\bar{X}}\right)$ Gandge et al. (1993) and Singh (1986)-type Estimator		0	1	α_1	0	1	η
$T_3 = \alpha_1\bar{y}_{st}\left(\frac{\bar{X}}{\bar{x}_{st}}\right)$ Kadilar and Cingi (2005) Estimator		1	0	α_1	0	1	η
$T_4 = \bar{y}_{st}\left(\frac{\bar{X}}{\bar{x}_{st}}\right)^{\phi}$ Kaur (1985) Estimator		1	0	1	0	ϕ	η
$T_5 = \bar{y}_{st}\left(\frac{\bar{x}_{st}}{\bar{X}}\right)^{\eta}$ Srivastava (1967) Estimator		1	0	0	1	ϕ	η
$T_6 = \bar{y}_{st}\left(\frac{\bar{X} + C_x}{\bar{x}_{st} + C_x}\right)$ Kadilar and Cingi (2003) Estimator		1	$\frac{C_x}{\bar{X}}$	1	0	1	η
$T_7 = \bar{y}_{st}\left(\frac{\bar{X} + \beta_2(x)}{\bar{x}_{st} + \beta_2(x)}\right)$ Kadilar and Cingi (2003) Estimator		1	$\frac{\beta_2(x)}{\bar{X}}$	1	0	1	η
$T_8 = \bar{y}_{st}\left(\frac{\bar{X}\beta_2(x) + C_x}{\bar{x}_{st}\beta_2(x) + C_x}\right)$ Kadilar and Cingi (2003) Estimator		$\beta_2(x)$	$\frac{C_x}{\bar{X}}$	1	0	1	η

5. Efficiency Comparison

In this section, we conducted a theoretical comparison of MSE's of certain common estimators existing in the literature and the instituted class of estimators.

(i) MSE $(\bar{y}_{st})$-MSE$(\bar{y}_{lrc})$>0

iff $\rho_c > 0$

(ii) MSE $(\bar{y}_{rc})$-MSE$(\bar{y}_{lrc})$>0

iff $\rho_c < \frac{R(RB-2E)}{A}$

(iii) MSE $(\bar{y}_{pc})$-MSE$(\bar{y}_{lrc})$>0

iff $\rho_c < \frac{R(RB+2E)}{A}$

(iv) MSE $(\bar{y}_{lrc})$-MSE$(\bar{y}_{RP}^{(C)})$>0

iff $\rho_c^2 < \left[\frac{\bar{U}^2}{A}\left\{1-\left(\frac{z_1 z_5^2+z_2 z_4^2-2z_3 z_4 z_5}{z_1 z_2-z_3^2}\right)\right\}\right]-1$

In part (i), (ii), and (iii) we have obtained the conditions under which the MSE of $\bar{y}_{lrc}$ is less than the MSE's of estimators $\bar{y}_{st}$, $\bar{y}_{rc}$, and $\bar{y}_{pc}$. And (iv) shows the conditions under which MSE of the proposed class of estimators $\bar{y}_{RP}^{(C)}$ is less than the MSE of $\bar{y}_{lrc}$ i.e. under which $\bar{y}_{RP}^{(C)}$ have outperformed $\bar{y}_{lrc}$ and so $\bar{y}_{st}$, $\bar{y}_{rc}$, and $\bar{y}_{pc}$.

6. Efficiency Comparison

To judge the merits of the proposed family of estimators over the usual unbiased estimator $\bar{y}_{st}$, we considered the following three population data sets whose statistics are described are as follows:

Population I: Singh and Mangat (1996), pp. 219

y: total number of milch cows in 1993 x: total number of milch cows in 1990

Table 1. Data set of population I

Total	Stratum	1	2	3
$N=24$	N_h	7	12	5
$n=10$	n_h	3	5	2
$\bar{X}=16.79167$	$\bar{X}_h$	15.285714	17.25	17.8
$\bar{Y}=19.16667$	$\bar{Y}_h$	17.428571	19.583333	20.6
$S_x^2=22.78079$	S_{xh}^2	20.904762	30.204545	10.700
$S_y^2=16.31884$	S_{yh}^2	17.619048	16.628788	13.300
$\rho=0.540819$	ρ_h	0.7654592	0.4066542	0.4945774
$R=1.14144$	γ_h	0.1904762	0.1166667	0.3000
$\rho_c=0.512755$	W_h^2	0.0850694	0.25000	0.0434028

Population: II Murthy (1967, p. 228)

y: factories in region; x: number of workers

Table 3. Data set of population II

Total	Stratum	1	2	3	4
$N=80$	N_h	25	23	16	16
$n=45$	n_h	14	13	9	9
$\bar{X}=284.75$	$\bar{X}_h$	71.0	140.696	362.937	749.5
$\bar{Y}=5182.64$	$\bar{Y}_h$	3156.64	4766.22	6334.19	7795.31
$S_x^2=270.495$	S_{xh}^2	14.6116	28.0364	91.3823	174.463
$S_y^2=1835.66$	S_{yh}^2	740.012	515.697	501.399	653.09
$\rho=0.9144$	ρ_h	0.8167	0.8231	0.9582	0.9805
$R=18.201$	γ_h	0.03143	0.03344	0.04861	0.04861
$\rho_c=0.67079$	W_h^2	0.09766	0.08266	0.04	0.04

Population: III Singh and Mangat (1996, pp. 218)

y: juice quantity x: weight of cane

Table 4. Data set of population III

Total	Stratum	1	2	3
$N=25$	N_h	6	12	7
$n=10$	n_h	3	4	3
$\bar{X}=326$	$\bar{X}_h$	366.666	310.833	317.143
$\bar{Y}=102.6$	$\bar{Y}_h$	135.00	99.166	80.714
$S_x^2=2700$	S_{xh}^2	2706.666	1881.06	2890.476
$S_y^2=558.583$	S_{yh}^2	80.00	226.515	120.238
$\rho=0.7314955$	ρ_h	0.9455626	0.948196	0.7532234
$R=0.314723$	γ_h	0.1666667	0.1666667	0.1904762
$\rho_c=0.8676778$	W_h^2	0.0576	0.2304	0.0784

For the purpose of the efficiency comparison of the members of the proposed family of estimators, we have computed the percent relative efficiencies (PREs) of the estimators with respect to the usual unbiased estimator $\bar{y}_{st}$ using the formula:

$$PRE\left(t,\bar{y}_{st}\right)=\frac{MSE\left(\bar{y}_{st}\right)}{MSE\left(t\right)}X100$$; where t is the considered estimators.

The findings are given in Table 1.

Table 5. PRE of the estimators w.r.t. $\bar{y}_{st}$

Estimators (t)	Population I	Population II	Population III
$\bar{y}_{st}$	100	100	100
$\bar{y}_{rc}$	67.95342	24.71606	192.7309
$\bar{y}_{pc}$	23.35866	9.0635	20.19156
$\bar{y}_{lrc}$	144.3298	415.8208	342.2806
$\bar{y}_{RP}^{(c)}$ at$(C_y, \rho_{yx}, C_x, 1, 1)$	172.5308	483.0426	2973.825
$\bar{y}_{RP}^{(c)}$ at$(0, \rho_{yx}, C_x, 1, 1)$	170.279	482.8143	2971.378
$\bar{y}_{RP}^{(c)}$ at$(1, \rho_{yx}, C_x, 1, 1)$	182.0956	483.4597	2982.78
$\bar{y}_{RP}^{(c)}$ at $(C_y, 1, C_x, 1, 1)$	**240.4512**	**722.5762**	**3058.676**

Table 6. PRE of some members of the proposed family of estimators for population I

a	c	d	η	ϕ	PRE
C_y	ρ_{yx}	C_x	1	1	172.5308
ρ_{yx}	C_x	$\rho_{yx}/\bar{X}$	1	1	588.0545
C_x	$C_x/\bar{X}$	ρ_{yx}	1	1	194.1695
$C_x/\bar{X}$	ρ_{yx}	C_x	1	1	170.4555
$\rho_{yx}/\bar{X}$	C_x	$C_x/\bar{X}$	1	1	805.1699
C_x	C_y	$C_x/\bar{X}$	1	1	729.9634
C_y	$C_x/\bar{X}$	C_x	1	1	182.7414
$\rho_{yx}/\bar{X}$	ρ_{yx}	$C_x/\bar{X}$	1	1	1481.509
0	ρ_{yx}	C_x	1	1	170.279
0	C_x	$\rho_{yx}/\bar{X}$	1	1	452.5473
0	$C_x/\bar{X}$	ρ_{yx}	1	1	194.3954
0	C_x	$C_x/\bar{X}$	1	1	783.7963
0	C_y	$C_x/\bar{X}$	1	1	603.5484
0	$C_x/\bar{X}$	C_x	1	1	182.9926
0	ρ_{yx}	$C_x/\bar{X}$	1	1	1405.744
1	ρ_{yx}	C_x	1	1	182.0956

1	C_x	$\rho_{yx}/\bar{X}$	1	1	800.5066
1	$C_x/\bar{X}$	ρ_{yx}	1	1	193.6055
1	C_x	$C_x/\bar{X}$	1	1	5551.977
1	C_y	$C_x/\bar{X}$	1	1	1623.469
1	$C_x/\bar{X}$	C_x	1	1	181.8123
C_y	1	C_x	1	1	240.4512
ρ_{yx}	1	$\rho_{yx}/\bar{X}$	1	1	9286.113
$C_x/\bar{X}$	1	C_x	1	1	233.6819
$\rho_{yx}/\bar{X}$	1	$C_x/\bar{X}$	1	1	2745.182
C_x	$C_x/\bar{X}$	1	1	1	201.3056
$\rho_{yx}/\bar{X}$	C_x	1	1	1	144.4556
C_x	C_y	1	1	1	150.881
C_y	$C_x/\bar{X}$	1	1	1	201.3421
ρ_{yx}	$C_y/\bar{Y}$	ρ_{yx}	1	1	199.3765
ρ_{yx}	$\rho_{yx}/\bar{Y}$	ρ_{yx}	1	1	185.103
0	$C_y/\bar{Y}$	ρ_{yx}	1	1	199.6869
0	$\rho_{yx}/\bar{Y}$	ρ_{yx}	1	1	185.6998
C_x	1	ρ_{yx}	1	1	171.2376
C_x	1	$C_x/\bar{X}$	1	1	15300.91

Table 7. PRE of some members of the proposed family of estimators for population II

a	c	d	η	ϕ	PRE
C_x	C_x	C_x	1	1	560.5276
C_x	ρ_{yx}	C_x	1	1	560.5276
C_y	ρ_{yx}	C_x	1	1	483.0426
$C_x/\bar{X}$	ρ_{yx}	C_x	1	1	482.8164
ρ_{yx}	$\rho_{yx}/\bar{Y}$	$C_x/\bar{X}$	1	1	442.8418
C_x	$C_x/\bar{X}$	ρ_{yx}	1	1	1258.973
C_x	$C_x/\bar{X}$	C_y	1	1	1101.764
C_y	$\rho_{yx}/\bar{Y}$	ρ_{yx}	1	1	1651.558
0	C_x	C_x	1	1	559.6518
0	ρ_{yx}	ρ_{yx}	1	1	559.6518
0	ρ_{yx}	C_x	1	1	482.8143

0	$\rho_{yx}/\bar{Y}$	$C_x/\bar{X}$	1	1	442.8912
0	$C_x/\bar{X}$	ρ_{yx}	1	1	1259.134
0	$C_x/\bar{X}$	C_y	1	1	1011.723
0	$\rho_{yx}/\bar{Y}$	ρ_{yx}	1	1	1351.46
1	C_x	C_x	1	1	560.5738
1	ρ_{yx}	ρ_{yx}	1	1	560.5738
1	ρ_{yx}	C_x	1	1	483.4597
1	$\rho_{yx}/\bar{Y}$	$C_x/\bar{X}$	1	1	442.8371
1	$C_x/\bar{X}$	ρ_{yx}	1	1	1259.851
1	$C_x/\bar{X}$	C_y	1	1	1011.686
1	$\rho_{yx}/\bar{Y}$	ρ_{yx}	1	1	1651.147
C_x	1	C_x	1	1	723.567
C_x	1	ρ_{yx}	1	1	934.5919
C_y	1	C_x	1	1	722.5762
$C_x/\bar{X}$	1	C_x	1	1	721.9942
C_y	1	ρ_{yx}	1	1	932.844
C_x	C_x	1	1	1	462.0076
ρ_{yx}	$\rho_{yx}/\bar{Y}$	1	1	1	1380.656
C_x	$C_x/\bar{X}$	1	1	1	1275.853
C_y	$\rho_{yx}/\bar{Y}$	1	1	1	1380.957

Table 8. PRE of some members of the proposed family of estimators for population III

a	C	d	η	ϕ	PRE
ρ_{yx}	C_x	C_y	1	1	9050.484
$C_y/\bar{Y}$	$C_y/\bar{Y}$	$C_y/\bar{Y}$	1	1	4817.493
C_y	ρ_{yx}	C_x	1	1	2973.825
$C_x/\bar{X}$	C_x	C_y	1	1	9360.872
C_x	C_y	$C_x/\bar{X}$	1	1	3907.847
$\rho_{yx}/\bar{Y}$	$\rho_{yx}/\bar{Y}$	$\rho_{yx}/\bar{Y}$	1	1	8416.968
$C_y/\bar{Y}$	$\rho_{yx}/\bar{Y}$	$\rho_{yx}/\bar{Y}$	1	1	4817.493
$C_x/\bar{X}$	$C_x/\bar{X}$	$C_x/\bar{X}$	1	1	4818.472
0	C_x	C_y	1	1	9361.087

0	$C_y/\bar{Y}$	$C_y/\bar{Y}$	1	1	4817.734
0	ρ_{yx}	C_x	1	1	2971.378
0	C_y	$C_x/\bar{X}$	1	1	3890.076
0	$\rho_{yx}/\bar{Y}$	$\rho_{yx}/\bar{Y}$	1	1	4817.734
0	$C_x/\bar{X}$	$C_x/\bar{X}$	1	1	4818.524
ρ_{yx}	C_x	0	1	1	4000.506
$C_y/\bar{Y}$	$C_y/\bar{Y}$	0	1	1	3914.693
C_y	ρ_{yx}	0	1	1	3940.849
$C_x/\bar{X}$	C_x	0	1	1	3914.494
C_x	C_y	0	1	1	3932.646
$\rho_{yx}/\bar{Y}$	$\rho_{yx}/\bar{Y}$	0	1	1	3915.247
$C_y/\bar{Y}$	$\rho_{yx}/\bar{Y}$	0	1	1	3914.693
$C_x/\bar{X}$	$C_x/\bar{X}$	0	1	1	3914.494
1	C_x	C_y	1	1	8942.166
1	$C_y/\bar{Y}$	$C_y/\bar{Y}$	1	1	4713.432
1	ρ_{yx}	C_x	1	1	2982.78
1	C_y	$C_x/\bar{X}$	1	1	4006.505
1	$\rho_{yx}/\bar{Y}$	$\rho_{yx}/\bar{Y}$	1	1	4713.432
1	$C_x/\bar{X}$	$C_x/\bar{X}$	1	1	4714.782
ρ_{yx}	1	C_y	1	1	2968.188
$C_y/\bar{Y}$	1	$C_y/\bar{Y}$	1	1	3888.937
C_y	1	C_x	1	1	3058.676
$C_x/\bar{X}$	1	C_y	1	1	2961.643
C_x	1	$C_x/\bar{X}$	1	1	3926.876
$\rho_{yx}/\bar{Y}$	1	$\rho_{yx}/\bar{Y}$	1	1	3835.804
$C_y/\bar{Y}$	1	$\rho_{yx}/\bar{Y}$	1	1	3835.293
$C_x/\bar{X}$	1	$C_x/\bar{X}$	1	1	3908.824
C_y	ρ_{yx}	1	1	1	8018.602

7. Cost Function for Problem Formulation

According to Khan *et al.* (2010), the survey's linear cost and fixed total cost C_0 are introduced as a function of $n_h; h = 1,2,\ldots,L$.

$$C_0 = \sum_{h=1}^{L} C_h\, n_h \tag{7.1}$$

where, C_h denotes the cost per unit of measuring each characteristic in the h stratum.; $h = 1,2,\dots,L$.
In order to approximate the fixed linear cost function's optimal mean square error, this is a particular method to express the optimization constraint:

$$\begin{aligned} &Minimize && MSE\left(t_{pr(st)}\right) \\ &\text{Subject to} && \textstyle\sum_{h=1}^{L} C_h\, n_h \le C_0 \\ & && 2 \le n_h \le N_h \end{aligned} \tag{7.2}$$

and $\quad n_h$ are integers; $h = 1,2,\dots,L$.

Using the cost function, compute the mean square error now.

$$MSE\left(t_{pr(st)}\right) = \sum_{h=1}^{L} W_h^2 \frac{1}{n_h}\left[1 + \alpha_1^2 Z_1 + \alpha_2^2 Z_2 + 2\alpha_1\alpha_1 Z_3 - 2\alpha_1 Z_4 - 2\alpha_2 Z_5\right]. \tag{7.3}$$

8. Applying Lagrange's multiplier method

Ito and Kunisch (2008) provided the Lagrange multiplier technique as an approach to attaining optimality under restricted conditions. They also discussed the conditions required for a minimum or maximum. The application of the Lagrange multiplier technique is able to identify the optimal value. The Lagrange multiplier function is expressed as

$$L(x,\lambda) = f(x) + \lambda g(x) \tag{7.4}$$

where L is the lagrangian, is the lagrange multiplier, g(x) is the constraint, f(x) is a function, and x is an integer value.

$$L(n_h,\lambda) = MSE\left(t_{pr(st)}\right) + \lambda\left(\sum_{h=1}^{L} C_h\, n_h - C_0\right) \tag{7.5}$$

The optimal values the estimators is

$$n_h = \frac{C_0\sqrt{\dfrac{W_h^2(Z)}{C_h}}}{\sum_{h=1}^{L}\sqrt{W_h^2(Z)C_h}}$$

where,

$$Z = 1 + \alpha_1^2 Z_1 + \alpha_2^2 Z_2 + 2\alpha_1\alpha_1 Z_3 - 2\alpha_1 Z_4 - 2\alpha_2 Z_5$$

$$Z_1 = 1 + \frac{A}{\bar{U}^2} + \left\{\phi^2 + \phi(\phi+1)\right\}\frac{c^2}{\bar{V}^2}B - 4\phi\frac{c}{\bar{U}\bar{V}}E$$

$$Z_2 = 1 + \frac{A}{\bar{U}^2} + \left\{\eta^2 - \eta(\eta-1)\right\}\frac{c^2}{\bar{V}^2}B + 4\eta\frac{c}{\bar{U}\bar{V}}E$$

$$Z_3 = 1 + \frac{A}{\bar{U}^2} - \left\{\phi\eta + \frac{\eta(\eta-1)}{2} + \frac{\phi(\phi+1)}{2}\right\}\frac{c^2}{\bar{V}^2}B + 2(\eta-\phi)\frac{c}{\bar{U}\bar{V}}E \; Z_4 = 1 + \left\{\frac{\phi(\phi+1)}{2}\right\}\frac{c^2}{\bar{V}^2}B - \phi\frac{c}{\bar{U}\bar{V}}E,$$

$$Z_5 = 1 - \left\{\frac{\eta(\eta-1)}{2}\right\}\frac{c^2}{\bar{V}^2}B - \eta\frac{c}{\bar{U}\bar{V}}E$$

$A = S_{yh}^2, B = S_{yh}^2$ and $E = S_{yxh}$.

9. Empirical study: Case Study II, Using Real Data

In this section, we use the real population, data taken from Yadav *et al* (2024). given by the data information is, the Indian census 2011 taken in the city of Lucknow, Uttar Pradesh. The study variable is the area in square kilometers within specific villages and auxiliary variable are number of households. The population is N = 645 and divided into four distinct city, non-overlapping. There are also summary data Tables 9 and Table 10 provided.

Table 9. Obtained results from real data

h	N_h	$\bar{Y}_h$	$\bar{X}_h$	ρ_h	C_{yh}^2	C_{xh}^2	C_h
1	237	116.236	148.886	0.996	0.315	0.367	2
2	164	307.603	406.878	0.996	0.184	0.133	3
3	90	547.444	727.316	0.978	1.642	1.357	4
4	154	757.100	1355.500	0.961	4.795	10.318	5

Table 10. Continue Obtained results from real data

C_{yh}	C_{xh}	S_{yh}	S_{yh}^2	S_{xh}	S_{xh}^2	S_{yxh}	S_{yxh}^2
0.561	0.606	65.222	4253.864	90.206	8137.055	0.339	0.115
0.429	0.365	131.937	17407.245	148.625	22089.297	0.156	0.024
1.282	1.165	701.593	492232.145	847.311	717936.402	1.460	2.132
2.190	3.212	1657.805	2748317.663	4354.066	18957887.308	6.759	45.688

10. Integer Programming Technique

The term "integer programming problem" means optimization constraints when variables have a problem with integer values. Mixed-variable problems arise when some of the variables are constant. The problem of integer linear programming arises from the assumption that all functions are linear. Following that, we use real data and a fixed linear cost function to compute the least mean square error possible. The optimization situation is then defined as

$$Minimize \quad \frac{160.013}{n_1}+\frac{360.057}{n_2}+\frac{561.926}{n_3}+\frac{1498.28}{n_4} \tag{7.12}$$

$$\text{Subject to} \quad \sum_{h=1}^{L} C_h\, n_h \leq C_0 \tag{7.13}$$

$$2 \leq n_h \leq N_h$$

$$\text{And} \qquad n_h \text{ are integers; } h = 1,2,3,4.$$

Table 11. The optimal values of the proposed estimators.

Estimators	n_1	n_2	n_3	n_4	n	MSE	PRE
$\bar{y}_{st}$	15	17	42	151	225	1370.229	100.000
$\bar{y}_{rc}$	14	15	37	160	222	1800.186	76.116
$\bar{y}_{pc}$	14	15	37	156	222	1800.195	76.116
$\bar{y}_{lrc}$	15	17	42	151	213	1586.537	86.366
$\bar{y}_{RP}^{(c)}$ at$(C_y, \rho_{yx}, C_x, 1{,}1)$	40	59	50	108	258	237.8473	576.096
$\bar{y}_{RP}^{(c)}$ at$(0, \rho_{yx}, C_x, 1, 1)$	40	60	51	107	258	1272.130	107.711
$\bar{y}_{RP}^{(c)}$ at$(1, \rho_{yx}, C_x, 1, 1)$	40	60	51	107	258	676.463	202.558
$\boldsymbol{\bar{y}_{RP}^{(c)}}$ **at** $\boldsymbol{(C_y, 1, C_x, 1, 1)}$	**40**	**60**	**51**	**107**	**258**	**35.355**	**3875.630**

11. Final comments

In this paper, we find out some members of the proposed family of estimators for different values of $(a, c, d, \alpha_1, \alpha_2, \phi, \eta)$ in Table 1. And as you can see in Table 5, the efficiency of the proposed estimator is the highest is 240.451, 722.576, and 3058.676, and the least PRE interval is [23.358 to 182.056], [9.064 to 483.459], and [20.192 to 2982.78] for population I, population II, and population III, respectively, and so on Table 6, Table 7, and Table 8, PRE of some members of the proposed family of estimators for spopulation I, population II, and population III, respectively. Therefore, under stratified random sampling with a linear cost function, the proposed estimate is the most efficient in the class of competing estimators, with the highest PRE being 3875.630 and the least PRE interval being [76.116 to 576.06]. We also found out the optimum value of the proposed estimators, which is in Table 11.

12. Conclusions

In this study, we developed an improved class of estimators for the population mean under stratified random sampling by applying variable transformations to both the study and auxiliary variables. Theoretical derivations of bias and mean squared error up to the first-order approximation, along with numerical analyses, showed that the proposed estimators are consistently more efficient and robust compared to traditional methods. The percentage relative efficiencies evaluated across multiple populations further confirmed their superior performance. Beyond theory, these estimators have practical applications in real-life scenarios. For instance, they can enhance the accuracy of crop yield predictions in agriculture, monitor patient recovery and treatment outcomes in clinical research, estimate disease prevalence in public health studies, evaluate supplier quality in manufacturing, and analyze large-scale social and economic survey data. Importantly, these benefits are achieved without increasing survey costs, making the proposed estimators a reliable and cost-effective tool for interdisciplinary research and applied statistics.

Declarations

Author Contributions

Conceptualization: YADAV, R.; SINGH, S.; SINGH, L.; MALIK, D.; VERMA, M. K.; YADAV, S. K. **Data curation:** YADAV, R.; SINGH, S.; SINGH, L.; MALIK, D.; VERMA, M. K.; YADAV, S. K. **Formal analysis:** YADAV, R.; SINGH, S.; SINGH, L.; MALIK, D.; VERMA, M. K.; YADAV, S. K. **Funding acquisition:** YADAV, R.; SINGH, S.; SINGH, L.; MALIK, D.; VERMA, M. K.; YADAV, S. K. **Investigation:** YADAV, R.; SINGH, S.; SINGH, L.; MALIK, D.; VERMA, M. K.; YADAV, S. K. **Methodology:** YADAV, R.; SINGH, S.; SINGH, L.; MALIK, D.; VERMA, M. K.; YADAV, S. K. **Project administration: Software:** YADAV, R.; SINGH, S.; SINGH, L.; MALIK, D.; VERMA, M. K.; YADAV, S. K. **Resources:** YADAV, R.; SINGH, S.; SINGH, L.; MALIK, D.; VERMA, M. K.; YADAV, S. K. **Supervision: Validation:** YADAV, R.; SINGH, S.; SINGH, L.; MALIK, D.; VERMA, M. K.; YADAV, S. K. **Visualization:** YADAV, R.; SINGH, S.; SINGH, L.; MALIK, D.; VERMA, M. K.; YADAV, S. K. **Writing - original draft:** YADAV, R.; SINGH, S.; SINGH, L.; MALIK, D.; VERMA, M. K.; YADAV, S. K. **Writing - review and editing:** YADAV, R.; SINGH, S.; SINGH, L.; MALIK, D.; VERMA, M. K.; YADAV, S. K.

Data Availability

The data that support the findings of this study are available in references.

Declaration of Competing Interest

The authors declare no conflict of interest.

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