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ARTICLE

A new generalization of the Remkan distribution and its application to survival data

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Abstract
Due to the increasing need for flexibility in baseline distribution, this study proposes a new generalization of the two-parameter Remkan distribution using the Alpha Power Transformed technique. Important statistical properties of the new distribution such as the reliability measures, the order statistics, moments and other related measures, and moment generating function were derived and the model parameters estimated using the maximum likelihood estimate technique. Finally, the flexibility of the new distribution was illustrated using two real life datasets and the results showed that the new Alpha Power Transformed Remkan distribution was the best amongst other competing three-parameter distributions.

Keywords: Alpha Power Transformed distributions; Lifetime distributions; Three-parameter distributions; Entropy; Reliability; Remkan distribution.

1. Introduction

In recent years, new two-parameter distributions have emerged in the literature. These new two-parameter distributions have been shown to provide a better fit to complex real-life datasets than the one-parameter distributions. Some of the recently developed two-parameter distributions include the Darna distribution (Shraa & Al-Omari, 2019), the Hamza distribution (Aijaz *et al.*, 2020), the Samade distribution (Aderoju, 2021), and the Alzoubi distribution (Benrabia & Alzoubi, 2021).

It is important to note that these distributions are a mixture of the Exponential and Gamma distributions. These two distributions are known to have their weaknesses. The weakness of the Exponential distribution is that the hazard rate function is constant; hence, it cannot handle datasets with monotone non-decreasing hazard rates (Elechi *et al.*, 2022; Epstein, 1958; Ronald *et al.*, 2011; Shukla, 2018b; Shukla, 2018). Furthermore, the weakness of the Gamma distribution is that the survival rate function cannot be expressed in closed form (Elechi *et al.*, 2022; Shanker, 2015a, 2015b). The weaknesses of these two distributions are what the aforementioned one-parameter and two-parameter distributions address, providing distributions whose survival rate function can be expressed in closed form and hazard rate functions capable of handling datasets with monotone non-decreasing hazard rates.

In contributing to this gap in the literature, Uwaeme and Akpan (2024) proposed a new two-parameter distribution called the Remkan distribution (Uwaeme & Akpan, 2024). The Remkan distribution is a three-component density of an Exponential (η), Gamma $(3,\eta)$, and Gamma $(4,\eta)$ distribution with mixing proportions $\pi_1 = \frac{\eta}{(\eta+2\phi+6)}$, $\pi_2 = \frac{2\phi}{(\eta+2\phi+6)}$, and $\pi_3 = \frac{6}{(\eta+2\phi+6)}$ such that

$$g(x;\eta,\phi) = \pi_1 g_1(x;\eta) + \pi_2 g_2(x;\eta) + \pi_3 g_3(x;\eta) \tag{1}$$

where $g_1(x;\eta) = \eta e^{-\eta x}$, $g_2(x;\eta) = \frac{\eta x^2 e^{-\eta x}}{\Gamma(3)}$, and $g_3(x;\eta) = \frac{\eta^2 x^3 e^{-\eta x}}{\Gamma(4)}$
therefore,

$$g(x_k;\eta,\phi) = \eta e^{-\eta x} \cdot \frac{\eta}{\eta+2\phi+6} + \frac{\eta x^2 e^{-\eta x}}{\Gamma(3)} \cdot \frac{2\phi}{\eta+2\phi+6} + \frac{\eta^2 x^3 e^{-\eta x}}{\Gamma(4)} \cdot \frac{6}{\eta+2\phi+6} \tag{2}$$

Solving equation 2 gives the probability density function (pdf) of the Remkan distribution

$$g(x;\eta,\phi) = \frac{\eta^2}{(\eta+2\phi+6)}[1 + \phi\eta x^2 + \eta^2 x^3]e^{-\eta x};\ x > 0, \eta > 0, \phi > 0 \tag{3}$$

The corresponding cumulative distribution function (cdf) of (3) is obtained as

$$G(x;\eta,\phi) = 1 - \left[1 + \frac{\phi\eta^3 x^3 + (3+\phi)\eta^2 x^3 + (6+2\phi)\eta x}{\eta+2\phi+6}\right] e^{-\eta x} \tag{4}$$

The authors introduced some of the statistical properties of the new distribution. They showed that the Remkan distribution exhibits shapes that are not bell-shaped, but positively skewed, unimodal, and right-tailed (Akpan & Uwaeme, 2024). Furthermore, Akpan (2024) addressed one of the weaknesses of the Remkan distribution which is its non-monotonic hazard rate function. The author applied the inverse transformation techniques to develop the Inverse Remkan distribution which addressed this weakness and also showed an improved flexibility using some real lifetime datasets (Akpan, 2024).

Another way of improving a baseline distribution is the introduction of an extra parameter to improve its overall flexibility. Over the years, several methods have been developed to improve the flexibility of a baseline distribution with the addition of an extra parameter. Some of these methods includes the skew normal distribution (Azzalini, 1985), the exponentiated family of distributions (Mudholkar & Srivastava, 1993), the Marshall-Olkin family of distributions (Marshall & Olkin, 1997), the beta family of distributions (Eugene *et al.*, 2002), the transmuted family of distributions (Shaw & Buckley, 2009), and the T-X family of distributions (Alzaatreh *et al.*, 2013) amongst others (Jones, 2015; Lee *et al.*, 2013).

Recently, Mahdavi and Kundu (2016) proposed a new family of distribution functions called the alpha power transformed (APT) family of distributions which adds an extra shape parameter to a baseline distribution (Mahdavi & Kundu, 2016). The pdf of the APT family of distributions is given by

$$f_{APT}(x) = \begin{cases} \frac{\log\tau}{\tau-1} g(x)\tau^{G(x)} & if\ \tau > 0, \tau \neq 1 \\ g(x) & if\ \ \tau = 1 \end{cases} \tag{5}$$

The corresponding cdf is given as

$$F_{APT}(x) = \begin{cases} \frac{\tau^{G(x)}-1}{\tau-1} & if\ \tau > 0,\ \tau \neq 1 \\ G(x) & if\ \tau = 1 \end{cases} \tag{6}$$

where $g(x)$ and $G(x)$ are the pdf and cdf of the baseline distribution.

This new family of distribution was shown to bring more flexibility to the baseline distribution. Another important characteristic of this new family is that it incorporates skewness to the family of distribution functions (Mead *et al.*, 2019). Since the introduction of the alpha power transformed family of distributions, several authors have adopted this approach to generalize some baseline distributions, improving flexibility, fits, and skewness. Some of these baseline distributions includes the extended exponential distribution (Hassan *et al.*, 2018), Weibull distribution (Nassar *et al.*, 2018), Lindley distribution (Dey *et al.*, 2019), Power Lindley distribution (Hassan *et al.*, 2019), Frechet distribution (Nasiru *et al.*, 2019), Inverse Lomax distribution (ZeinEldin *et al.*, 2020), Pareto distribution (Sakthivel, 2020), the Weibull-G family of distributions (Elbatal *et al.*, 2021), the Garima distribution (Mohiuddin *et al.*, 2021), the Kumaraswamy distribution (Hozaien *et al.*, 2021), Rama distribution (Mohiuddin & Kannan, 2021a), the Aradhana distribution (Mohiuddin & Kannan, 2021b), Quasi Aradhana distribution (Mohiuddin *et al.*, 2022), Pranav distribution (Mohiuddin *et al.*, 2022), Sujatha distribution (Mohiuddin & Kannan, 2022), Power Ailamujia distribution (Gomaa *et al.*, 2023)

From the foregoing therefore, the motivation of this paper is to propose a new generalization of the Remkan distribution using the alpha power transformed technique called the Alpha Power Transformed Remkan distribution and its statistical properties. The subsequent sections of the paper will be arranged as follows. Section 2 discusses the new alpha power transformed Remkan distribution with the derivation of the pdf, the cdf, and their plots, section 3 discusses the mathematical properties of the Alpha Power Transformed Remkan distribution as well as the plots of the risk measurement function to highlight the shape, and section 4 concludes the paper with some remarks.

2. The Alpha Power Transformed Remkan distribution

This section will introduce the pdf and the cdf of the Alpha Power Transformed Remkan distribution and illustrate the different shapes of the Alpha Power Transformed Remkan distribution. The new Alpha Power Transformed Remkan distribution is obtained by putting equations 3 and 4 into equations 5 and 6 respectively.

Proposition 1: If a random variable X follows the Alpha Power Transformed Remkan distribution (APTR) with parameters η, ϕ, and τ then the pdf and cdf are respectively given by

$$f_{APTR}(x;\eta,\phi,\tau) = \frac{\eta^2 \log\tau}{(\tau-1)(\phi+\eta)}[1+\phi\eta x^2+\eta^2 x^3]e^{-\eta x} \cdot \tau^{1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}};\ x > 0, \eta > 0, \phi > 0, \tau \neq 1 \tag{7}$$

and

$$F_{APTR}(x;\eta,\phi,\tau) = \frac{\tau^{1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}}-1}{\tau-1};\ x>0, \eta>0, \phi>0, \tau>0, \tau\neq 1 \quad (8)$$

The Alpha Power Transformed Remkan distribution derived above is denoted by $APTR(\eta,\phi,\tau)$. The graphical plots of the theoretical density and distribution function (for some selected but different real points of η, ϕ, and τ) of the Alpha Power Transformed Remkan distribution are shown in the Figure 1 and Figure 2 below.

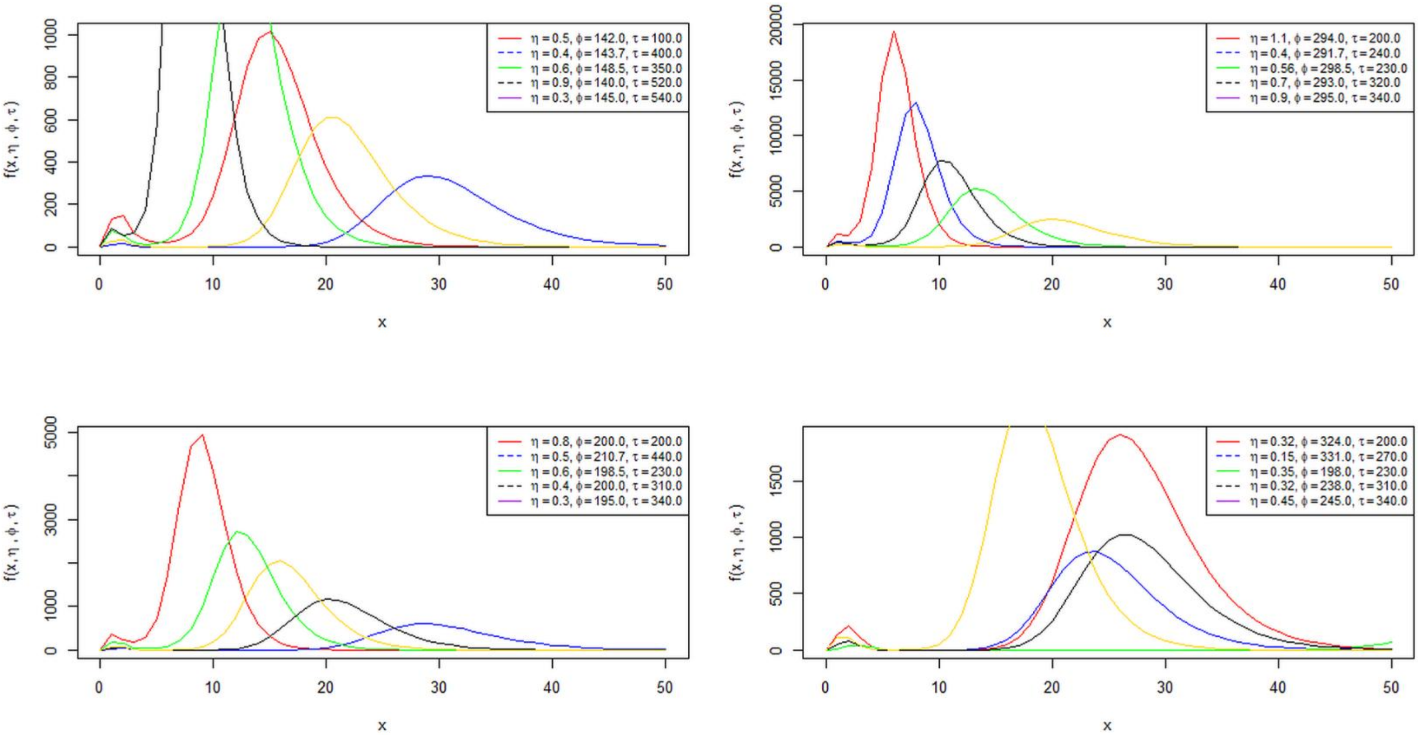

Figure 1. The graphical plots of the probability density function (for some selected but different real points of η, ϕ, *and* τ) of an Alpha Power Transformed Remkan distribution.

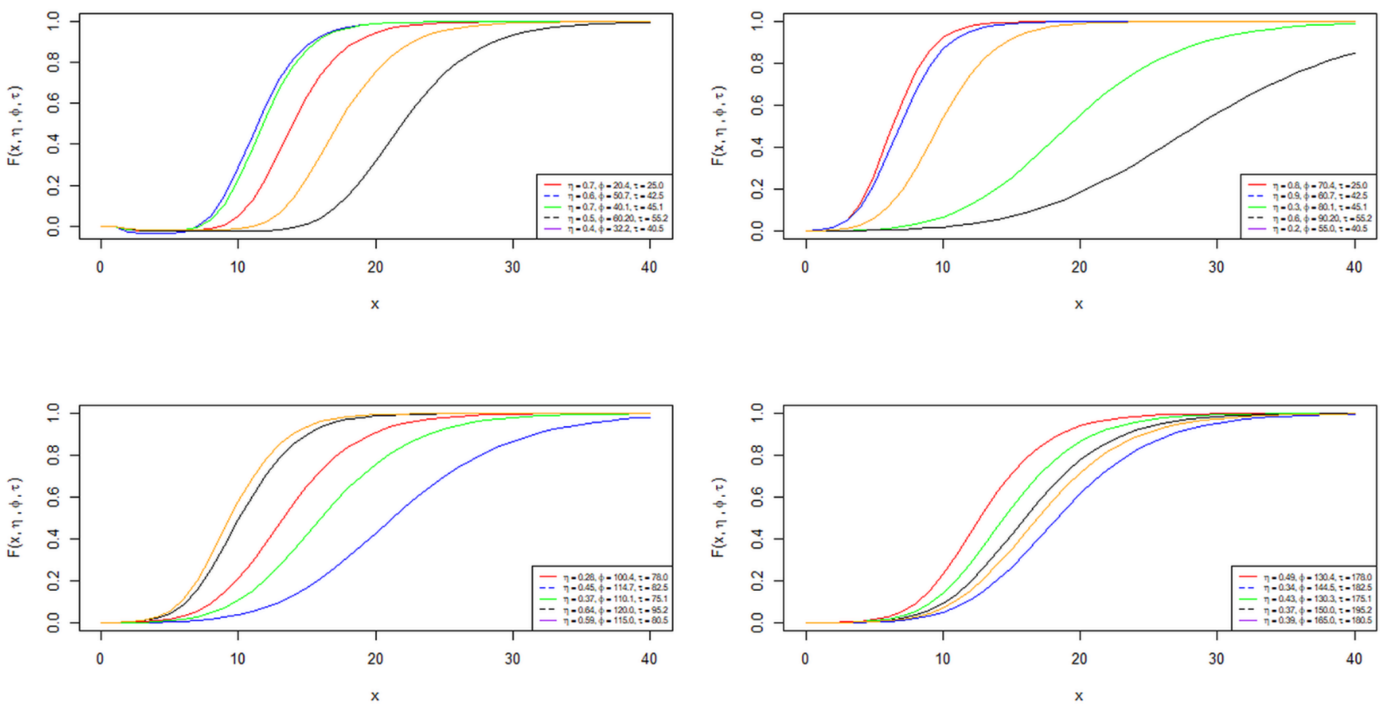

Figure 2. The graphical plots of the cumulative distribution function (for some selected but different real points of η, ϕ, *and* τ) of an Alpha Powr Transformed Remkan distribution.

The curves displayed in Figure 1 are not bell-shaped, but are positively skewed, unimodal, and right tailed. In addition, the curve shows that increasing the value of ϕ leads to a considerable increase in the peak of the curve. In addition, the curves displayed in Figure 2 shows that the cumulative distribution function converges to one.

3. Mathematical Properties of the Alpha Power Transformed Remkan Distribution

In this section, we derive and present some of the mathematical properties of the Alpha Power Transformed Remkan distribution such as the moment, moment generating function, and characteristic function.

3.1 Moments of the Alpha Power Transformed Remkan Distribution

We derive the rth moment of the Alpha Power Transformed Remkan distribution in this subsection.

Theorem 1

Given a random variable X, following the Alpha Power Transformed Remkan distribution, the rth order moment about origin, $E(X^r)$ of the Alpha Power Transformed Remkan distribution is given by

$$\mu_r' = E(X^r) = Q\left[\frac{\Gamma(r+2j+k-2l+1)}{(\eta+\eta i)^{(r+2j+k-2l+1)}} + \frac{\phi\eta\Gamma(r+2j+k-2l+3)}{(\eta+\eta i)^{(r+2j+k-2l+3)}} + \frac{\eta^2\Gamma(r+2j+k-2l+4)}{(\eta+\eta i)^{(r+2j+k-2l+4)}}\right] \quad (9)$$

where $Q = \left(\frac{\tau\log\tau}{\tau-1}\right)\frac{\phi^{j+k}2^l(3+\phi)^k\eta^{2+3j-k-l}}{(\eta+2\phi+6)^{j+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\tau)^i}{i!}$

Proof:

The rth crude or uncorrected moments of a random variable can be written as

$$\mu_r' = E(X^r) = \int_0^\infty x^r f_{APTR}(x;\eta,\phi,\tau)dx \tag{10}$$

$$= \int_0^\infty x^r \cdot \frac{\eta^2 \log\tau}{(\tau-1)(\eta+2\phi+6)}[1+\phi\eta x^2+\eta^2 x^3]e^{-\eta x}\cdot \tau^{1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}} dx \tag{11}$$

$$= \frac{\eta^2\tau \log\tau}{(\tau-1)(\eta+2\phi+6)}\int_0^\infty x^r \cdot [1+\phi\eta x^2+\eta^2 x^3]e^{-\eta x}\cdot \tau^{-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}} \tag{12}$$

Using the power series expansion $\tau^{-z} = \sum_{i=0}^\infty \frac{(-\log\tau)^i}{i!} z^i$, we obtain

$$E(X^r) = \frac{\eta^2\tau \log\tau}{(\tau-1)(\eta+2\phi+6)}\sum_{i=0}^\infty \frac{(-\log\tau)^i}{i!}\int_0^\infty x^r \cdot [1+\phi\eta x^2+\eta^2 x^3]\cdot \left(\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]\right)^i e^{-(\eta+\eta i)x}dx \tag{13}$$

Using the binomial series expansion $(1+\tau)^n = \sum_{k=0}^n \binom{n}{k}\tau^k$, and after some simplification we have,

$$E(X^r) = \left(\frac{\tau\log\tau}{\tau-1}\right)\frac{2^l\phi^{j+k}(3+\phi)^k\eta^{2+3j-k-l}}{(\eta+2\phi+6)^{j+1}}\sum_{i=0}^\infty\sum_{j=0}^i\sum_{k=0}^j\sum_{l=0}^k\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\tau)^i}{i!}\int_0^\infty x^{r+2j+k-2l}[1+\phi\eta x^2+\eta^2x^3]e^{-(\eta+\eta i)x}dx \tag{14}$$

Let $Q = \left(\frac{\tau\log\tau}{\tau-1}\right)\frac{2^l\phi^{j+k}(3+\phi)^k\eta^{2+3j-k-l}}{(\eta+2\phi+6)^{j+1}}\sum_{i=0}^\infty\sum_{j=0}^i\sum_{k=0}^j\sum_{l=0}^k\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\tau)^i}{i!}$. Hence,

$$E(X^r) = Q\left[\int_0^\infty x^{r+2j+k-2l}e^{-(\eta+\eta i)x}dx + \phi\eta\int_0^\infty x^{r+2j+k-2l+2}e^{-(\eta+\eta i)x}dx + \eta^2\int_0^\infty x^{r+2j+k-2l+3}e^{-(\eta+\eta i)x}dx\right] \tag{15}$$

Recall $\int_0^\infty z^w e^{-\varrho z}\,dz = \frac{\Gamma(w+1)}{\varrho^{w+1}}$, then

$$\therefore E(X^r) = Q\left[\frac{\Gamma(r+2j+k-2l+1)}{(\eta+\eta i)^{(r+2j+k-2l+1)}} + \frac{\phi\eta\Gamma(r+2j+k-2l+3)}{(\eta+\eta i)^{(r+2j+k-2l+3)}} + \frac{\eta^2\Gamma(r+2j+k-2l+4)}{(\eta+\eta i)^{(r+2j+k-2l+4)}}\right] \tag{16}$$

Which completes the proof.

In particular, the first four (4) moments about the origin of the Alpha Power Transformed Remkan distribution is obtained by substituting the values of $r = 1, 2, 3, 4$ as follows;

$$\mu_1' = E(X) = Q\left[\frac{\Gamma(2j+k-2l+2)}{(\eta+\eta i)^{(2j+k-2l+2)}} + \frac{\phi\eta\Gamma(2j+k-2l+4)}{(\eta+\eta i)^{(2j+k-2l+4)}} + \frac{\eta^2\Gamma(2j+k-2l+5)}{(\eta+\eta i)^{(2j+k-2l+5)}}\right] \tag{17}$$

$$\mu_2' = E(X^2) = Q\left[\frac{\Gamma(2j+k-2l+3)}{(\eta+\eta i)^{(2j+k-2l+3)}} + \frac{\phi\eta\Gamma(2j+k-2l+5)}{(\eta+\eta i)^{(2j+k-2l+5)}} + \frac{\eta^2\Gamma(2j+k-2l+6)}{(\eta+\eta i)^{(2j+k-2l+6)}}\right] \tag{18}$$

$$\mu_3' = E(X^3) = Q\left[\frac{\Gamma(2j+k-2l+4)}{(\eta+\eta i)^{(2j+k-2l+4)}} + \frac{\phi\eta\Gamma(2j+k-2l+6)}{(\eta+\eta i)^{(2j+k-2l+6)}} + \frac{\eta^2\Gamma(2j+k-2l+7)}{(\eta+\eta i)^{(2j+k-2l+7)}}\right] \quad (19)$$

$$\mu_4' = E(X^4) = Q\left[\frac{\Gamma(2j+k-2l+5)}{(\eta+\eta i)^{(2j+k-2l+5)}} + \frac{\phi\eta\Gamma(2j+k-2l+6)}{(\eta+\eta i)^{(2j+k-2l+6)}} + \frac{\eta^2\Gamma(2j+k-2l+8)}{(\eta+\eta i)^{(2j+k-2l+8)}}\right] \quad (20)$$

3.2 Moment generating function

Here, we propose the moment generating function for the Alpha Power Transformed Remkan distribution on this subsection.

Theorem 2

Given a random variable X, following the Alpha Power Transformed Remkan distribution, the moment generating function of X, $M_X(t)$ of the Alpha Power Transformed Remkan distribution is given by

$$M_X(t) = E(e^{tx}) = Q\left[\frac{\Gamma(m+2j+k-2l+1)}{(\eta+\eta i)^{(r+2j+k-2l+1)}} + \frac{\phi\eta\Gamma(m+2j+k-2l+3)}{(\eta+\eta i)^{(r+2j+k-2l+3)}} + \frac{\eta^2\Gamma(m+2j+k-2l+4)}{(\eta+\eta i)^{(r+2j+k-2l+4)}}\right] \quad (21)$$

where $Q = \left(\frac{\tau\log\tau}{\tau-1}\right)\frac{2^l\phi^{j+k}(3+\phi)^k\eta^{2+3j-k-l}}{(\eta+2\phi+6)^{j+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{\infty}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\tau)^i}{i!}\frac{t^r}{r!}$

Proof:

The moment generating function of X, $M_X(t)$ can be written as

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f_{APTR}(x;\eta,\phi,\tau)dx \quad (22)$$

Recall that $e^{tx} = \sum_{k=0}^{\infty}\frac{(tx)^k}{k!}$. Substituting, we obtain

$$M_X(t) = E(e^{tx}) = \int_0^\infty \sum_{r=0}^{\infty}\frac{(tx)^r}{r!} f_{APTR}(x;\eta,\phi,\tau)dx \quad (23)$$

$$M_X(t) = E(e^{tx}) = \sum_{r=0}^{\infty}\frac{(t)^r}{r!}\int_0^\infty x^r f_{APTR}(x;\eta,\phi,\tau)dx \quad (24)$$

$$M_X(t) = E(e^{tx}) = \sum_{r=0}^{\infty}\frac{(t)^r}{r!}E(X^r) \quad (25)$$

$$M_X(t) = \left(\frac{\tau\log\tau}{\tau-1}\right)\frac{2^l\phi^{j+k}(3+\phi)^k\eta^{2+3j-k-l}}{(\eta+2\phi+6)^{j+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{\infty}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\tau)^i}{i!}\frac{t^r}{r!}\left[\frac{\Gamma(m+2j+k-2l+1)}{(\eta+\eta i)^{(m+2j+k-2l+1)}} + \frac{\phi\eta\Gamma(m+2j+k-2l+3)}{(\eta+\eta i)^{(m+2j+k-2l+3)}} + \frac{\eta^2\Gamma(m+2j+k-2l+4)}{(\eta+\eta i)^{(m+2j+k-2l+4)}}\right] \quad (26)$$

$$\therefore M_X(t) = E(e^{tx}) = Q\left[\frac{\Gamma(m+2j+k-2l+1)}{(\eta+\eta i)^{(r+2j+k-2l+1)}} + \frac{\phi\eta\Gamma(m+2j+k-2l+3)}{(\eta+\eta i)^{(r+2j+k-2l+3)}} + \frac{\eta^2\Gamma(m+2j+k-2l+4)}{(\eta+\eta i)^{(r+2j+k-2l+4)}}\right] \quad (27)$$

where $Q = \left(\frac{\tau\log\tau}{\tau-1}\right)\frac{2^l\phi^{j+k}(3+\phi)^k\eta^{2+3j-k-l}}{(\eta+2\phi+6)^{j+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{\infty}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\tau)^i}{i!}\frac{(it)^r}{r!}$

Which completes the proof.

Similarly, the characteristics function of the Alpha Power Transformed Remkan distribution is obtained by

$$\varphi_X(t) = M_X(it) \tag{28}$$

$$\varphi_X(t) = \left(\frac{\tau\log\tau}{\tau-1}\right)\frac{2^l\phi^{j+k}(3+\phi)^k\eta^{2+3j-k-l}}{(\eta+2\phi+6)^{j+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{\infty}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\tau)^i}{i!}\frac{(it)^r}{r!}\left[\frac{\Gamma(m+2j+k-2l+1)}{(\eta+\eta i)^{(r+2j+k-2l+1)}}+\frac{\phi\eta\Gamma(m+2j+k-2l+3)}{(\eta+\eta i)^{(r+2j+k-2l+3)}}+\frac{\eta^2\Gamma(m+2j+k-2l+4)}{(\eta+\eta i)^{(r+2j+k-2l+4)}}\right] \tag{29}$$

$$\therefore \varphi_X(t) = M_X(it) = T\left[\frac{\Gamma(m+2j+k-2l+1)}{(\eta+\eta i)^{(r+2j+k-2l+1)}}+\frac{\phi\eta\Gamma(m+2j+k-2l+3)}{(\eta+\eta i)^{(r+2j+k-2l+3)}}+\frac{\eta^2\Gamma(m+2j+k-2l+4)}{(\eta+\eta i)^{(r+2j+k-2l+4)}}\right] \tag{30}$$

where $T = \left(\frac{\tau\log\tau}{\tau-1}\right)\frac{2^l\phi^{j+k}(3+\phi)^k\eta^{2+3j-k-l}}{(\eta+2\phi+6)^{j+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{\infty}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\tau)^i}{i!}\frac{(it)^r}{r!}$

4. Statistical Properties of the Alpha Power Transformed Remkan Distribution

In this section, we derive and present some statistical properties of the Alpha Power Transformed Remkan distribution. These includes the survivorship or existence measurement function, risk measurement function, order statistics, and entropy.

4.1 Order statistics

We derive the rth moment of the Alpha Power Transformed Remkan distribution in this sub.

Theorem 3

Given a continuous random variable X, following the Alpha Power Transformed Remkan distribution, the pdf and cdf of the pth order statistics, say $X = X_{(p)}$, is given respectively by

$$f_{APTR}(x) = \frac{n!\,\eta^2[1+\phi\eta x^2+\eta^2x^3]e^{-\eta x}(\tau\log\tau)}{(\eta+2\phi+6)(p-1)!\,(n-p)!\,(\tau-1)}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i\,\tau^{-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}}\left[\frac{\tau^{1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}}-1}{\tau-1}\right]^{p+i-1} \tag{31}$$

and

$$F_{APTR}(x) = \sum_{j=p}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\binom{n-j}{i}(-1)^i\left[\frac{\tau^{1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}}-1}{\tau-1}\right]^{j+1} \tag{32}$$

Proof:

Given a continuous random variable X, the pdf of the pth order statistics, say $X = X_{(p)}$, is obtained by

$$f_{APTR}(x) = \frac{n!g(x_k;\Phi)}{(p-1)!(n-p)!}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i\, G(x_k;\Phi)^{p+i-1} \tag{33}$$

$$f_{APTR}(x) = \frac{n!\left(\frac{\eta^2\log\tau}{(\tau-1)(\eta+2\phi+6)}[1+\phi\eta x^2+\eta^2x^3]e^{-\eta x}\cdot\tau^{1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}}\right)}{(p-1)!\,(n-p)!}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i \left[\frac{\tau^{1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}}-1}{\tau-1}\right]^{p+i-1} \tag{34}$$

$$f_{APTR}(x) = \frac{n!\,\eta^2[1+\phi\eta x^2+\eta^2x^3]e^{-\eta x}(\tau\log\tau)}{(\eta+2\phi+6)(p-1)!\,(n-p)!\,(\tau-1)}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i\,\tau^{-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}} \left[\frac{\tau^{1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}}-1}{\tau-1}\right]^{p+i-1} \tag{35}$$

which completes the proof.

Correspondingly, given a continuous random variable Y, the cdf of the pth order statistics, say $Y = Y_{(p)}$, is obtained by

$$F_{APTR}(x) = \sum_{j=p}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\binom{n-j}{i}(-1)^i G(x_k;\Phi)^{j+1} \tag{36}$$

$$F_{APTR}(x) = \sum_{j=p}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\binom{n-j}{i}(-1)^i\left[\frac{\tau^{1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}}-1}{\tau-1}\right]^{j+1} \tag{37}$$

Which completes the proof.

4.2 Entropy

We derive the rth moment of the Alpha Power Transformed Remkan distribution in this sub.

Entropy measures the uncertainties associated with a random variable of a probability distributions. Shannon (Shannon, 1951) and Rényi's entropy (Rényi, 1961) are widely used in the literature.

Theorem 4:

Given a random variable Y, which follows the Alpha Power Transformed Remkan distribution $f_{APTR}(x;\eta,\phi,\tau)$. The Rényi entropy is given by

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\left(\frac{\tau\log\tau}{\tau-1}\right)^{\gamma}\frac{2^l\phi^{j+k+m-n}(3+\phi)^k\eta^{2\gamma+3j-k-l+m+n}}{(\eta+2\phi+6)^{\gamma+j}}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{\gamma}\sum_{n=0}^{m}\binom{i}{j}\binom{j}{k}\binom{k}{l}\binom{\gamma}{m}\binom{m}{n}\frac{\gamma^i(-\log\tau)^i}{i!}\left[\frac{\Gamma(3j+k-2l+2m+n+1)}{\eta(i+\gamma)^{(3j+k-2l+2m+n+1)}}\right]\right]$$

(38)

Proof:
The Rényi entropy is given by

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\int_0^\infty f_{APTR}^{\gamma}(x_k;\Phi)dx\right] \tag{39}$$

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\int_0^\infty \left(\frac{\eta^2\log\tau}{(\tau-1)(\eta+2\phi+6)}[1+\phi\eta x^2+\eta^2x^3]e^{-\eta x}\cdot \tau^{1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}}\right)^{\gamma} dx\right] \tag{40}$$

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\left(\frac{\tau\log\tau}{\tau-1}\right)^{\gamma}\frac{\eta^{2\gamma}}{(\eta+2\phi+6)^{\gamma}}\int_0^\infty[1+\phi\eta x^2+\eta^2x^3]^{\gamma}e^{-\gamma\eta x}\tau^{1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]\gamma e^{-\eta x}}dx\right] \tag{41}$$

Applying the power series expansion $\tau^{-z} = \sum_{i=0}^{\infty}\frac{(-\log\tau)^i}{i!}z^i$, we obtain

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\left(\frac{\tau\log\tau}{\tau-1}\right)^{\gamma}\frac{\eta^{2\gamma}}{(\eta+2\phi+6)^{\gamma}}\sum_{i=0}^{\infty}\frac{(-\log\tau)^i}{i!}\int_0^\infty[1+\phi\eta x^2+\eta^2x^3]^{\gamma}e^{-\gamma\eta x}\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]^i\gamma^i e^{-\eta i x}dx\right] \tag{42}$$

Using the binomial series expansions $(1+\tau)^n = \sum_{k=0}^{n}\binom{n}{k}\tau^k$, and after some simplification we have,

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\left(\frac{\alpha\log\alpha}{\alpha-1}\right)^{\gamma}\frac{2^l\phi^{j+k+m-n}(3+\phi)^k\eta^{2\gamma+3j-k-l+m+n}}{(\eta+2\phi+6)^{\gamma+j}}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{\gamma}\sum_{n=0}^{m}\binom{i}{j}\binom{j}{k}\binom{k}{l}\binom{\gamma}{m}\binom{m}{n}\frac{\gamma^i(-\log\tau)^i}{i!}\int_0^\infty x^{3j+k-2l+2m+n}e^{-\eta(i+\gamma)x}dx\right] \tag{43}$$

$$\therefore T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\left(\frac{\tau\log\tau}{\tau-1}\right)^{\gamma}\frac{2^l\phi^{j+k+m-n}(3+\phi)^k\eta^{2\gamma+3j-k-l+m+n}}{(\eta+2\phi+6)^{\gamma+j}}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{\gamma}\sum_{n=0}^{m}\binom{i}{j}\binom{j}{k}\binom{k}{l}\binom{\gamma}{m}\binom{m}{n}\frac{\gamma^i(-\log\tau)^i}{i!}\left[\frac{\Gamma(3j+k-2l+2m+n+1)}{\eta(i+\gamma)^{(3j+k-2l+2m+n+1)}}\right]\right] \tag{44}$$

Which completes the proof.

4.3 Reliability Indices

Given any probability distribution, the reliability analysis is always considered based on the Existence Measurement Function and Risk Measurement Function. Hence, for the Alpha Power

Transformed Remkan distribution, the Existence Measurement Function and Risk Measurement Function is given below.

4.3.1 Existence Measurement Function

The existence measurement function (also known as survival function) is defined as the probability that an item does not fail prior to some time t (Elechi *et al.*, 2022; Epstein, 1958; Ronald *et al.*, 2011; Shanker & Shukla, 2020).

The existence measurement function of the Alpha Power Transformed Remkan distribution is given by

$$S_{APTR}(x) = 1 - F_{APTR}(x; \eta, \phi, \tau) \tag{45}$$

$$S_{APTR}(x) = 1 - \left[\frac{\tau^{1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}} - 1}{\tau - 1}\right] \tag{46}$$

$$S_{APTR}(x) = \frac{\tau - \tau^{1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}}}{\tau - 1}; x > 0, \eta > 0, \phi > 0, \tau > 0, \tau \neq 1 \tag{47}$$

4.3.2 Risk Measurement Function

The risk measurement function (also known as hazard rate function) on the other hand can be seen as the conditional probability of failure, given it has survived to the time t (Elechi *et al.*, 2022; Ronald *et al.*, 2011; Shanker, 2016; Umeh & Ibenegbu, 2019).

The risk measurement function of the Alpha Power Transformed Remkan distribution is given by

$$h_{APTC}(x) = \frac{f_{APTC}(x_k; \Phi)}{1 - F_{APTC}(x_k; \Phi)} \tag{48}$$

$$h_{APTR}(x) = \frac{\frac{\eta^2 \log\tau}{(\tau-1)(\eta+2\phi+6)}[1+\phi\eta x^2+\eta^2x^3]e^{-\eta x}.\tau^{1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}}}{1-\left[\frac{\tau^{1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}} - 1}{\tau-1}\right]} \tag{49}$$

$$h_{APTR}(x) = \frac{\frac{\eta^2 \log\alpha}{(\eta+2\phi+6)}[1+\phi\eta x^2+\eta^2x^3]e^{-\eta x}.\tau^{1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}}}{\tau-\tau^{1-\left[1+\frac{\phi\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}}}; x > 0, \eta > 0, \phi > 0, \tau > 0, \tau \neq 1 \tag{50}$$

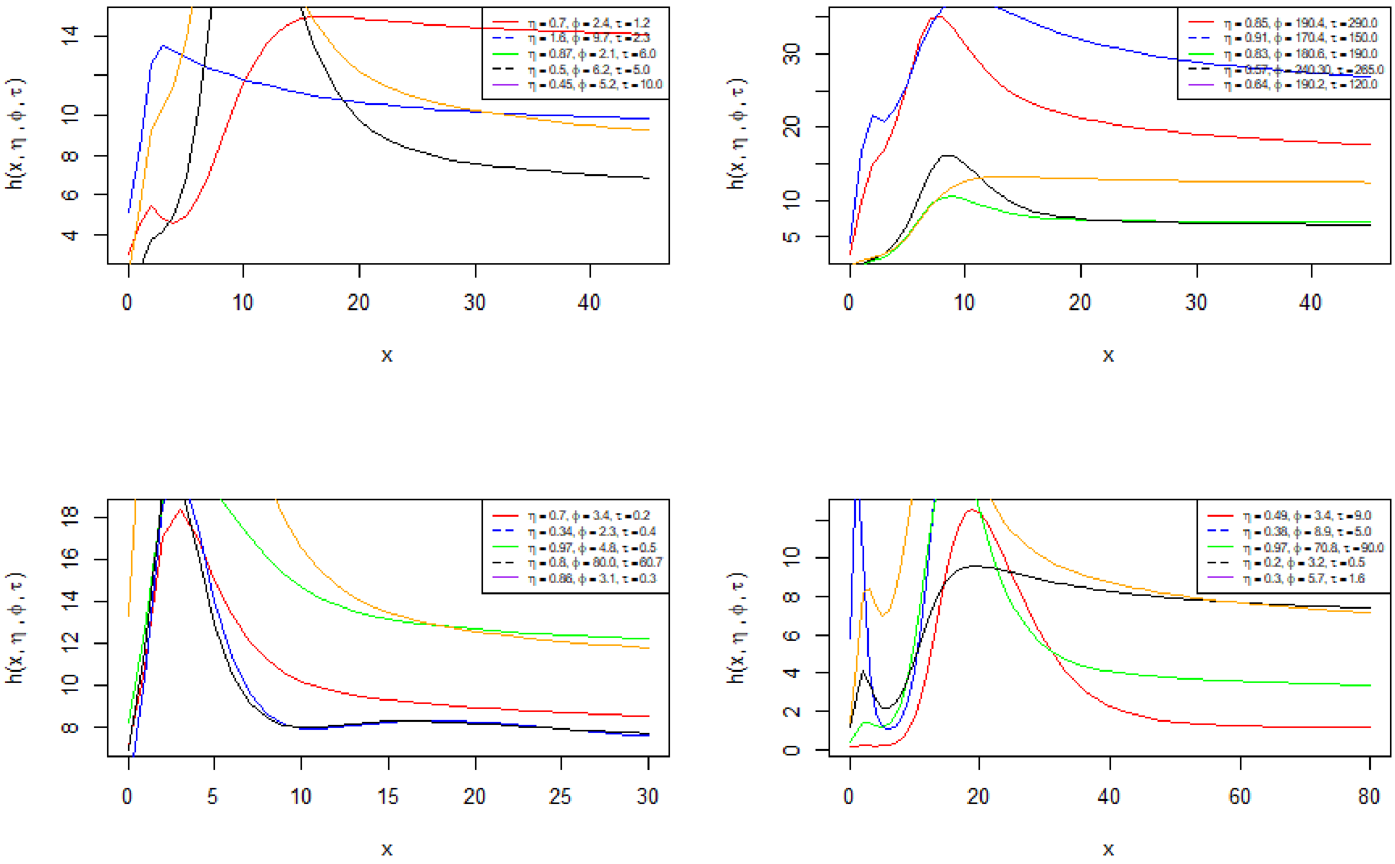

Figure 3. The graphical plots of the risk measurement function (for some selected but different real points of η, ϕ, *and* τ) of an Alpha Power Transformed Remkan distribution.

4.4 Parameter Estimation

Let $Y_1, Y_2, Y_3, \ldots, Y_m$ be a random sample of size m from the Alpha Power Transformed Remkan distribution $g(y_k; \eta, \phi, \tau)$. Then the log-likelihood function of parameters is given by

$$LL(y_1, y_2, y_3, \ldots, y_m; \Phi) = \prod_{i=1}^{m} \ln g(y_k; \Phi) \tag{51}$$

$$L(y; \eta, \phi, \tau) = \prod_{i=1}^{m} \left[\frac{\eta^2 \log \tau}{(\tau-1)(\eta+2\phi+6)} [1 + \phi\eta y^2 + \eta^2 y^3] e^{-\eta y} \cdot \tau^{1-\left[1+\frac{\phi\eta^3 y^3+(3+\phi)\eta^2 y^3+(6+2\phi)\eta y}{\eta+2\phi+6}\right] e^{-\eta y}} \right] \tag{52}$$

$$L(y; \eta, \phi, \tau) = \frac{(\eta^2 \tau \log \tau)^m}{(\tau-1)^m (\eta+2\phi+6)^m} \prod_{i=1}^{m} \left[[1 + \phi\eta y_i^2 + \eta^2 y_i^3] e^{-\eta y_i} \cdot \tau^{-\left[1+\frac{\phi\eta^3 y_i^3+(3+\phi)\eta^2 y_i^3+(6+2\phi)\eta y_i}{\eta+2\phi+6}\right] e^{-\eta y_i}} \right] \tag{53}$$

The log-likelihood function is given by;

$$LL(y;\eta,\phi,\tau) = m[\ln(\ln\tau) - \ln(\tau-1) + \ln(\tau) + 2\ln(\eta) - \ln(\eta+2\phi+6)] + \sum_{i=1}^{m}\ln(1+\phi\eta y_i^2+\eta^2 y_i^3) - \eta y_i - \ln(\tau)\sum_{i=1}^{m}\left(1+\frac{\phi\eta^3 y_i^3+(3+\phi)\eta^2 y_i^3+(6+2\phi)\eta y_i}{\eta+2\phi+6}\right)e^{-\eta y_i} \quad (54)$$

In order to maximize the log-likelihood, we solve the nonlinear likelihood equations obtained from the partial differentiation of (54) w.r.t τ, η *and* ϕ as shown below;

$$\frac{\partial LL}{\partial \tau} = \frac{m}{\tau\log\tau} + \frac{m}{\tau} - \frac{m}{\tau-1} - \frac{1}{\tau}\sum_{i=1}^{m}\left(1+\frac{\phi\eta^3 y_i^3+(3+\phi)\eta^2 y_i^3+(6+2\phi)\eta y_i}{\eta+2\phi+6}\right)e^{-\eta y_i} \quad (55)$$

$$\frac{\partial LL}{\partial \eta} = \frac{2m}{\eta} - \frac{m}{(\eta+2\phi+6)} - \sum_{i=1}^{m} y_i + \sum_{i=1}^{m}\left[\frac{2\eta y_i^3+\phi y_i^2}{1+\phi\eta y_i^2+\eta^2 y_i^3}\right] - \frac{\left[\frac{3\phi\eta^2 y_i^2+2(3+\phi)\eta y_i^3+(6+2\phi)y_i}{\eta+2\phi+6} - \frac{\phi\eta^3 y_i^2+(3+\phi)\eta^2 y_i^3+(6+2\phi)\eta y_i}{(\eta+2\phi+6)^2}\right]e^{-\eta y_i}}{\left[1+\frac{\phi\eta^3 y_i^2+(3+\phi)\eta^2 y_i^3+(6+2\phi)\eta y_i}{\eta+2\phi+6}\right]} + \ln\sum_{i=1}^{m}\left[\tau\left(1+\frac{\phi\eta^3 y_i^3+(3+\phi)\eta^2 y_i^3+(6+2\phi)\eta y_i}{\eta+2\phi+6}\right)\right]e^{-\eta y_i}\sum_{i=1}^{m} y_i \quad (56)$$

$$\frac{\partial LL}{\partial \phi} = -\frac{2m}{(\eta+2\phi+6)} + \frac{\eta y_i^2}{1+\phi\eta y_i^2+\eta^2 y_i^3} - \frac{\left[\frac{\eta^3 y_i^2+\eta^2 y_i^3+2\eta y_i}{\eta+2\phi+6} - \frac{2\left(\phi\eta^3 y_i^2+(3+\phi)\eta^2 y_i^3+(6+2\phi)\eta y_i\right)}{(\eta+2\phi+6)^2}\right]}{\left[1+\frac{\phi\eta^3 y_i^2+(3+\phi)\eta^2 y_i^3+(6+2\phi)\eta y_i}{\eta+2\phi+6}\right]} \quad (57)$$

In order to obtain the estimates of the parameters using the nonlinear equations above, we equate equations to zero and solve simultaneously. The solutions cannot be solved analytically. Hence, we solve numerically using the MaxLik package of in the R software (Toomet *et al.*, 2015) with "BFGS" algorithm.

5. Application

This section discusses the flexibility and superiority of the Alpha Power Transformed Remkan Distribution (APTR) to some competing distributions using two real life uncensored data sets. This dataset is from Bekker *et al.* (2000) which corresponds to the survival times (in years) of a group of patients given chemotherapy treatment alone. The data consisting of survival times (in years) for 45 patients. The dataset is shown in Table 1 below.

Table 1. Survival times (in years) for patients receiving chemotherapy treatment alone (n = 45 observations)

0.047	0.115	0.121	0.132	0.164	0.197	0.203	0.260	0.282
0.296	0.334	0.395	0.458	0.466	0.501	0.507	0.529	0.534
0.540	0.641	0.644	0.696	0.841	0.863	1.099	1.219	1.271
1.326	1.447	1.485	1.553	1.581	1.589	2.178	2.343	2.416
2.444	2.825	2.830	3.578	3.658	3.743	3.978	4.003	4.033

The second dataset consists of 101 observations of stress-rupture life of kevlar 49/epoxy strands which are subjected to constant sustained pressure at the 90% stress level until all have failed, so that the complete data set with the exact times of failure is recorded. These failure times in hours, are originally given by Barlow & Proschan (1981)

The dataset is shown in Table 2 below.

Table 2. Stress-rupture life measurements (in thousands of hours) for 101 Kevlar 49/epoxy strand specimens

0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.04	0.05
0.06	0.07	0.07	0.08	0.09	0.09	0.10	0.10	0.11
0.11	0.12	0.13	0.18	0.19	0.20	0.23	0.24	0.24
0.29	0.34	0.35	0.36	0.38	0.40	0.42	0.43	0.52
0.54	0.56	0.60	0.60	0.63	0.65	0.67	0.68	0.72
0.72	0.72	0.73	0.79	0.79	0.80	0.80	0.83	0.85
0.90	0.92	0.95	0.99	1.00	1.01	1.02	1.03	1.05
1.10	1.10	1.11	1.15	1.18	1.20	1.29	1.31	1.33
1.34	1.40	1.43	1.45	1.50	1.51	1.52	1.53	1.54
1.54	1.55	1.58	1.60	1.63	1.64	1.80	1.80	1.81
2.02	2.05	2.14	2.17	2.33	3.03	3.03	3.34	4.20
4.69	7.89.							

These dataset are then fitted with the Alpha Power Transformed Remkan Distribution (APTR) and compared with the Alpha Power within Weibull Quantile distribution (APWQ) (Nassar *et al.*, 2018), and Alpha Power Transformed Copoun distribution (APTC) with corresponding pdfs.

$$g_{APWQ}(x;\eta) = \frac{(\tau-1)\eta\phi x^{\phi-1}e^{-\eta x^{\phi}}}{\log\tau\left(1+(\tau-1)\left(1-e^{-\eta x^{\phi}}\right)\right)} \tag{58}$$

$$g_{APTC}(x;\eta,\phi,\alpha) = \frac{\eta^2\log\alpha}{(\alpha-1)(\phi+\eta)}\left[1+\frac{\phi\eta^2x^3}{6}\right]e^{-\eta x}\cdot\alpha^{1-\left[1+\frac{\phi\eta^3x^3+\phi\eta^2x^3+\phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x}} \tag{59}$$

This comparison is done using some measures for testing the goodness of fit of a distribution. The measures used are the parameter estimates, the log likelihood, the Akaike Information Criteria (AIC) and the Bayesian Information Criteria (BIC) -2lnL, Akaike Information Criterion (AIC) (Club, 2016), Bayesian Information Criterion (BIC) (Pollock *et al.*, 1999), Consistent Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC). In general, the smaller the values of AIC, BIC, CAIC, and HQIC the better the fit to the data.

$$AIC = 2k - 2\ln L \tag{60}$$

$$BIC = k\ln n - 2\ln L \tag{61}$$

$$CAIC = AIC + \frac{2k(k+1)}{(n-k-1)} \tag{62}$$

$$HQIC = 2k\ln(\ln(n)) - 2\ln L \tag{63}$$

where k is the number of parameters, n is the sample size of the dataset, and L is the likelihood function.

Table 3. Goodness of fit for the Uncensored strengths of glass fibres

Distribution	MLE's	S.E	-2ln L	AIC	BIC	CAIC	HQIC
APTR	$\hat{\eta}=1.093$	**0.0185**	**106.1209**	**112.1209**	**113.7342**	**112.4066**	**108.7944**
	$\hat{\phi}=1.333$	**0.0471**					
	$\hat{\tau}=1.000$	**0.0189**					
APWQ	$\hat{\eta}=0.1215$	0.0329	112.9882	118.9882	120.6015	119.5736	115.6617
	$\hat{\phi}=1.8759$	0.1791					
	$\hat{\tau}=123.20$	4.8525					
APTC	$\hat{\eta}=1.1013$	0.0192	153.9403	159.9403	161.5537	160.5257	156.6138
	$\hat{\phi}=1.5191$	0.0592					
	$\hat{\tau}=1.000$	0.0206					

The parameter estimates and their goodness of fit of the different models for the first dataset are presented in Table 3. From the results, the Alpha Power Transformed Remkan Distribution (APTR) performed better than the competing distributions.

Table 4. Goodness of fit for the Uncensored breaking stress of carbon fibres in (Gba)

Distribution	MLE's	S.E	-2ln L	AIC	BIC	CAIC	HQIC
APTR	$\hat{\eta}=1.2435$	**0.010**	**132.190**	**138.190**	**141.420**	**138.312**	**135.248**
	$\hat{\phi}=1.2678$	**0.022**					
	$\hat{\tau}=1.000$	**0.011**					
APWQ	$\hat{\eta}=0.9879$	0.895	205.997	211.997	215.227	212.245	209.056
	$\hat{\phi}=0.9280$	0.272					
	$\hat{\tau}=1.000$	3.083					
APTC	$\hat{\eta}=1.2431$	0.010	238.637	244.637	247.868	244.760	241.696
	$\hat{\phi}=1.2928$	0.023					
	$\hat{\tau}=1.000$	0.011					

The parameter estimates and their goodness of fit of the different models for the second dataset are presented in Table 4. From the results, the Alpha Power Transformed Remkan distribution (APTR) performed better than the competing distributions.

6. Conclusions

This paper proposed a new three-parameter distribution known as the Alpha Power Transformed Remkan (APTR) distribution. The new distribution is a generalization of the Remkan distribution using the Alpha Power Transformed technique. The mathematical and statistical properties of the Alpha Power Transformed Remkan distribution such as the moments, moment generating functions, order statistics, entropy, and reliability indices was derived and presented. The properties of the new Alpha Power Transformed Remkan distribution showed that it can be used to model lifetime datasets with unimodal, positively skewed, and right tailed properties. Furthermore, the risk measurement function of the Alpha Power Transformed Remkan distribution can model datasets with decreasing failure rate (DFR) and L-Shaped hazard rate in survival analysis. In addition, the method of maximum likelihood estimate was adopted to derive the estimates of the parameters. The flexibility of the new Alpha Power Transformed Remkan distribution was compared with other competing distributions using two different real life datasets. The results obtained showed that the new Alpha Power Transformed Remkan distribution gave the best fit to the data based on some model selection criteria. Hence, the new Alpha Power Transformed Remkan distribution is therefore recommended as an alternative to other existing distributions.

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Conflicts of Interest

The authors declare no conflict of interest.

Author Contributions

Conceptualization: UWAEME, O. R. **Data curation:** UWAEME, O. R.; OBASI, C. K. **Formal analysis:** UWAEME, O. R.; SYLVA, L. **Investigation:** UWAEME, O. R. **Methodology:** UWAEME, O. R.; NWAGWU, G. C. **Software:** UWAEME, O. R.; OBASI, C. K. **Resources:** NWAGWU, G. C.; SYLVA, L. **Visualization:** UWAEME, O. R. **Writing - original draft:** UWAEME, O. R. **Writing - review and editing:** OBASI, C. K.; NWAGWU, G. C.; SYLVA, L.

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