Brazilian Journal of Biometrics, **44**, 1-14 (2026)
doi:10.28951/bjb.v44i3.862

BRAZILIAN JOURNAL OF BIOMETRICS
ISSN:2764-5290

ARTICLE

Estimation of the exponential memory-type ratio estimator using simple random sampling

Bhatt Ravi Jitendrakumar[1] , Monika Saini[1] and Ashish Kumar*[1]

[1]Department of Mathematics and Statistics, Manipal University Jaipur, Jaipur, India
*Corresponding author. Email: ashishbarak2020@gmail.com;

(Received: March 14, 2025; Revised: March 10, 2026; Accepted: March 18, 2026; Published: August 26, 2026)

Section editor: Douglas Silva.

Abstract

It is significant in survey sampling that the population mean is estimated with the highest possible accuracy. A new memory-type exponential-ratio estimator is defined and its performance under simple random sampling (SRS) is evaluated in this research. The proposed estimator is shown to utilize past auxiliary information very effectively which results in a significantly lower mean square error (MSE) than other existing estimators. Based on theoretical derivations and genuine datasets, the proposed estimator was tested, and it was found to continually outperform traditional ratio-type estimators. Empirical results, including those from the comparisons of precision relative efficiency (PRE), confirm its superiority using reduced variance and increased efficiency. The results obtained indicate that the use of memory type modifications in exponential-ratio estimators enables the attainment of more accurate estimates of the population mean in sample surveys.

Keywords: Simple random sampling; Mean square error; Auxiliary variable; Exponential-ratio estimator; Memory-type estimator; Survey sampling.
Mathematics Subject Classification (2020): 62D05, 62G05.

1. Introduction

In statistical inference, particularly with estimating parameters of a population with the least amount of error, survey sampling is of importance. The precision of estimations is achieved through the integration of auxiliary information, hence ratio-type estimators are common within use. An example of advancement in this area would be the inventing of the so-called memory type estimators, which improved estimation of population parameters by recalling information from previous samples. Previously, the idea of memory-type estimator was posited to increase the efficacy of conventional estimators by using older auxiliary information. (Cochran, 1940) and (Srivastava, 1967) is credited with aiding the advancement of auxiliary information in sample surveys, which introduced modifications like exponential-ratio estimators (Bahl & Tuteja, 1991). (Murthy, 1964), (Sisodia and Dwivedi, 1981), (Kadilar and Cingi, 2003 & 2004), (Singh *et al.*

[1] © Brazilian Journal of Biometrics. This is an open access article distributed under the terms of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/)

2004), (Shabbir and Gupta, 2007), (Singh *et al.*, 2008), (Yadav *et al.*, 2015), (Zaman *et al.*, 2021), (Zaman *et al.*, 2022), (Irfan *et al.*, 2022), (Riyaz *et al.*, 2022), (Saini *et al.*, 2022), (Kumar *et al.*, 2023) and (Saini *et al.*, 2024) have participated in covering means to lighten the mean squared error through more estimators using auxiliary information and variable since. The goal of this research is to offer better approaches to help with advanced memory-type properties of simple random sampling in a broad, refined, weighted exponential-ratio estimator. Ratio estimators have shown great accuracy in the use of auxiliary variables. The incorporation of memory-type features into exponential-ratio types can exceptionally reduce estimation errors and improve precision.

By using past and current information, (Robert, 1959) used very first time EWMA statistic which is also called memory type statistic in control chart test under simple random sampling, which inspire the author (Muhammad Noor-ul-Amin, 2019) to proposed memory type ratio and product estimator using simple random sampling scheme. This paper describes a new memory-type exponential-ratio estimator, based on previous works of (Muhammad Noor-ul-Amin, 2019), and it assesses performance theoretically. The derivation of the estimator and its comparison with classical estimators prove its superiority in reducing MSE and improving efficiency. A real dataset is used to test its effectiveness, where estimates from this new memory-type exponential-ratio estimator are compared with those from other existing estimators. The rest of the paper is organized as follows: Existing estimators and their properties are reviewed in Section 2. The proposed estimator is introduced in section 3 with theoretical properties in Section 4. In Section 5 Numerical illustration from real data is provided. Results are discussed in Section 7, and conclusions with key findings and implications for future research are also given in same.

2. Material, methods & existing estimators

Let N be the finite population with selecting sample size n using SRSWOR. Let y be the study variable and x be the auxiliary variable with sample mean of $\bar{y}$ and $\bar{x}$ respectively. Let $\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}$ and $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$ both are corresponding to the population mean $\bar{Y} = \frac{\sum_{i=1}^{N} y_i}{N}$ and $\bar{X} = \frac{\sum_{i=1}^{N} x_i}{N}$ respectively.

We define the following error terms in order to obtain the bias and MSE of the suggested estimators up to the first order of approximation in SRS.

Let $e_y = \frac{\bar{y} - \bar{Y}}{\bar{Y}}$ and $e_x = \frac{\bar{x} - \bar{X}}{\bar{X}}$ such that,

$$E(e_y) = E(e_x) = 0;$$

$$E(e_y^2) = \vartheta\left(\frac{\lambda}{2-\lambda}\right) C_y^2;\ E(e_x^2) = \vartheta\left(\frac{\lambda}{2-\lambda}\right) C_x^2;$$

$$E(e_y\, e_x) = \vartheta\left(\frac{\lambda}{2-\lambda}\right) C_{yx};$$

$$Where, C_{yx} = \rho_{yx}\, C_y\, C_x;\ \vartheta = \left[\frac{1}{n} - \frac{1}{N}\right];$$

$$S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2,\ S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{X})^2\ and\ S_{yx} = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})(x_i - \bar{X}).$$

We may use a range of existing estimators, such as the unbiased estimator, ratio estimator, and exponential estimator for population mean to estimate the performance of our suggested estimator in simple random sampling. These estimators are represented as follows.

The population mean estimate (Unbiased estimator) of the study variable that is most frequently utilized is provided by,

$$\widehat{Y}_0 = \bar{y}$$

$$\widehat{Y}_0 = \sum_{i=1}^{n} \frac{y_i}{n}$$

An unbiased estimator of population mean, sample mean is provided up to the first order of approximation by,

$$MSE\left(\widehat{Y}_0\right) = \vartheta \bar{Y}^2 C_y^2 \quad (1)$$

Where, $\vartheta = \frac{1-f}{n}, f = \frac{n}{N}, C_y^2 = \frac{S_y^2}{\bar{Y}^2}$

The definition of the ratio estimator for population mean estimation is given by (Cochran, 1940),

$$\widehat{Y}_{cr} = \bar{X}\left(\frac{\bar{y}}{\bar{x}}\right)$$

The bias and MSE are given below,

$$Bias\left(\widehat{Y}_{cr}\right) = \vartheta \bar{Y}\left[C_x^2 - C_x\, C_y\, \rho\right]$$

$$MSE\left(\widehat{Y}_{cr}\right) = \vartheta \bar{Y}^2\left[C_y^2 + C_x^2 - 2\rho C_y\, C_x\right] \quad (2)$$

When the study and the auxiliary variable have a negative association, (Murthy, 1964) recommended using the product technique of estimation.

$$\widehat{Y}_{Mu} = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)$$

$$Bias\left(\widehat{Y}_{Mu}\right) = \vartheta \bar{Y}\left[\rho C_y\, C_x\right]$$

$$MSE\left(\widehat{Y}_{Mu}\right) = \vartheta \bar{Y}^2\left[C_y^2 + C_x^2 + 2\rho C_y\, C_x\right] \quad (3)$$

In order to calculate the population mean the product exponential estimator was introduced by (Bahl and Tuteja, 1991) and is defined as,

$$\widehat{Y}_{pe} = \bar{y}\, exp\left(\frac{\bar{x} - \bar{X}}{\bar{x} + \bar{X}}\right)$$

The bias and MSE to first degree of approximation are

$$Bias\left(\widehat{Y}_{pe}\right) = \vartheta \bar{Y}\left[\frac{1}{2}\rho C_y C_x - \frac{1}{8} C_x^2\right]$$

$$MSE\left(\widehat{Y}_{pe}\right) = \vartheta \bar{Y}^2\left[C_y^2 + \frac{1}{4} C_x^2 + \rho C_y C_x\right] \quad (4)$$

A ratio type estimator was presented by (Bahl and Tuteja, 1991) to estimate the population mean, which is defined as,

$$\widehat{Y}_{re} = \bar{y}\, exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right)$$

The bias and MSE to first degree of approximation are

$$Bias\left(\widehat{Y}_{re}\right) = \vartheta \bar{Y}\left[\frac{3}{8} C_x^2 - \frac{1}{2}\rho C_y C_x\right]$$

$$MSE\left(\widehat{Y}_{re}\right) = \vartheta \bar{Y}^2\left[C_y^2 + \frac{1}{4} C_x^2 - \rho C_y C_x\right] \quad (5)$$

A population mean estimator of the ratio type was presented by (Kadilar and Cingi, 2004),

$$\widehat{Y}_{1Pr} = \left(\frac{\bar{y} + b(\bar{X} - \bar{x})}{\bar{x}}\right)\bar{X}$$

where $\bar{X}$ and $\bar{x}$ represent the study variable's sample and population means, respectively. As a biased estimator, $\widehat{Y}_{1Pr}$ is defined by its bias and mean square error, up to the first order of approximation,

$$Bias\left(\widehat{Y}_{1Pr}\right) = \frac{\vartheta}{\overline{Y}}\left[R_1^2 S_x^2\right]$$

$$MSE\left(\widehat{Y}_{1Pr}\right) = \vartheta\left[R_x^2 S_x^2 + S_y^2 (1-\rho^2)\right] \quad (6)$$

Where $R_x = \frac{\overline{Y}}{\overline{X}}$

(Kadilar and Cingi, 2004) proposed a ratio type estimator of population mean

$$\widehat{Y}_{SD} = \left(\frac{\bar{y} + b(\bar{X} - \bar{x})}{\bar{x} + C_x}\right)(\bar{X} + C_x)$$

where $\bar{X}$ and $\bar{x}$ represent the study variable's sample and population means, respectively. As a biased estimator, $\widehat{Y}_{SD}$ is defined by its bias and mean square error, up to the first order of approximation,

$$Bias\left(\widehat{Y}_{SD}\right) = \frac{\vartheta}{\overline{Y}}\left[R_{SDx}^2 S_x^2\right]$$

$$MSE\left(\widehat{Y}_{SD}\right) = \vartheta\left[R_{SDx}^2 S_x^2 + S_y^2 (1-\rho^2)\right] \quad (7)$$

Where $R_{SDx} = \frac{\overline{Y}}{\overline{X} + C_x}$

(Kadilar and Cingi, 2004) suggested another ratio type estimator of population mean

$$\widehat{Y}_{SK} = \left(\frac{\bar{y} + b(\bar{X} - \bar{x})}{\bar{x} + \beta_2}\right)(\bar{X} + \beta_2)$$

where $\bar{X}$ and $\bar{x}$ represent the study variable's sample and population means, respectively. As a biased estimator, $\widehat{Y}_{SK}$ is defined by its bias and mean square error, up to the first order of approximation,

$$Bias\left(\widehat{Y}_{SK}\right) = \frac{\vartheta}{\overline{Y}}\left[R_{SKx}^2 S_x^2\right]$$

$$MSE\left(\widehat{Y}_{SK}\right) = \vartheta\left[R_{SKx}^2 S_x^2 + S_y^2 (1-\rho^2)\right] \quad (8)$$

Where $R_{SKx} = \frac{\overline{Y}}{\overline{X} + \beta_2}$

A population mean estimator of the ratio type was recommended by (Kadilar and Cingi, 2004),

$$\widehat{Y}_{US1} = \left(\frac{\bar{y} + b(\bar{X} - \bar{x})}{\bar{x}\beta_2 + C_x}\right)(\bar{X}\beta_2 + C_x)$$

where $\bar{X}$ and $\bar{x}$ represent the study variable's sample and population means, respectively. As a biased estimator, $\widehat{Y}_{US1}$ is defined by its bias and mean square error, up to the first order of approximation,

$$Bias\left(\widehat{Y}_{US1}\right) = \frac{\vartheta}{\overline{Y}}\left[R_{US1x}^2 S_x^2\right]$$

$$MSE\left(\widehat{Y}_{US1}\right) = \vartheta\left[R_{US1x}^2 S_x^2 + S_y^2 (1-\rho^2)\right] \quad (9)$$

Where $R_{US1x} = \frac{\overline{Y}}{\bar{x}\beta_2 + C_x}$

To estimate population mean (Kadilar and Cingi, 2004) proposed a ratio type estimator of population mean

$$\widehat{Y}_{US2} = \left(\frac{\bar{y} + b(\bar{X} - \bar{x})}{\bar{x}C_x + \beta_2}\right)(\bar{X}C_x + \beta_2)$$

where $\bar{X}$ and $\bar{x}$ represent the study variable's sample and population means, respectively. As a biased estimator, $\widehat{Y}_{US2}$ is defined by its bias and mean square error, up to the first order of approximation,

$$Bias(\widehat{Y}_{US2}) = \frac{\vartheta}{\overline{Y}}[R_{US2x}^2 S_x^2]$$
$$MSE(\widehat{Y}_{US2}) = \vartheta[R_{US2x}^2 S_x^2 + S_y^2(1-\rho^2)] \quad (10)$$

Where $R_{US2x} = \frac{\overline{Y}}{\bar{x}C_x + \beta_2}$

In order to increase the efficiency of the estimators, the EWMA statistic was first utilized by (Roberts, 1959) to combine past current data.

$$\widehat{\widehat{Y}}_Z^m = \lambda\bar{y} + (1-\lambda)Z_{i-1}$$

where $\bar{y}$ is the mean of current sample, λ is the smoothing constant, which is the weight given to the observations. λ value has range from 0 to 1 in which smaller values is indicating that we are giving weight to the past data similarly larger value indicating weight is given to the current data.

The MSE is as follows:

$$MSE\left(\widehat{\widehat{Y}}_Z^m\right) = \vartheta\left(\frac{\lambda}{2-\lambda}\right)\overline{Y}^2 C_y^2 \quad (11)$$

(Muhammad 2019) used EWMA statistic which is a memory type statistic that utilized current and past information to propose the ratio type estimator in simple random sampling.

$$\widehat{\widehat{Y}}_{rmi}^m = \bar{x}\left(\frac{Z_i}{Q_i}\right)$$

The bias and MSE are given below,

$$Bias\left(\widehat{\widehat{Y}}_{rmi}^m\right) = \vartheta\left(\frac{\lambda}{2-\lambda}\right)\overline{Y}[C_x^2 - C_x C_y \rho]$$
$$MSE\left(\widehat{\widehat{Y}}_{rmi}^m\right) = \vartheta\left(\frac{\lambda}{2-\lambda}\right)\overline{Y}^2[C_y^2 + C_x^2 - 2\rho C_y C_x] \quad (12)$$

3. Proposed estimator & its properties

We have created a modified version of the exponential memory type ratio estimator, inspired by (Srivastava, 1967).

$$\widehat{\widehat{Y}}_{prop}^m = Z\ exp\left[\frac{\overline{X}-Q}{\overline{X}+Q}\right]^\alpha \quad (13)$$

Z and Q are the sample means of the study and auxiliary variables, respectively.

$\overline{X}$ and $\overline{Y}$ are the population means of the auxiliary and study variables.

α is a constant parameter.

Now expanding the equation (13) using Taylor series of expansion, we get

$$\widehat{\widehat{Y}}_{prop}^m = \overline{Y}(1+e_y)\ exp\left[\frac{\overline{X}-\overline{X}(1+e_x)}{\overline{X}+\overline{X}(1+e_x)}\right]^\alpha$$
$$\widehat{\widehat{Y}}_{prop}^m = \overline{Y}(1+e_y)\ exp\left[\frac{-e_x}{2+e_x}\right]^\alpha$$
$$\widehat{\widehat{Y}}_{prop}^m = \overline{Y}(1+e_y)\left[1 - \frac{\alpha e_x}{2} + \frac{\alpha e_x^2}{4} + \frac{\alpha^2 e_x^2}{8}\right]$$
$$\widehat{\widehat{Y}}_{prop}^m = \overline{Y}\left[1 - \frac{\alpha e_x}{2} + \frac{\alpha e_x^2}{4} + \frac{\alpha^2 e_x^2}{8} + e_y - \frac{\alpha e_x e_y}{2}\right]$$
$$\widehat{\widehat{Y}}_{prop}^m = \overline{Y}\left[1 + e_y - \frac{\alpha e_x}{2} + \frac{\alpha e_x^2}{4} + \frac{\alpha^2 e_x^2}{8} - \frac{\alpha e_x e_y}{2}\right]$$
$$\left(\widehat{\widehat{Y}}_{prop}^m - \overline{Y}\right) = \overline{Y}\left[e_y - \frac{\alpha e_x}{2} + \frac{\alpha e_x^2}{4} + \frac{\alpha^2 e_x^2}{8} - \frac{\alpha e_x e_y}{2}\right] \quad (14)$$

Take the expectation on both the sides of equation (14), we get

$$E\left(\widehat{\bar{Y}}_{prop}^{m}-\bar{Y}\right)=\bar{Y}\left[E\left(e_{y}\right)-\frac{\alpha}{2}E\left(e_{x}\right)+\frac{\alpha}{4}E\left(e_{x}^{2}\right)+\frac{\alpha^{2}}{8}\left(e_{x}^{2}\right)-\frac{\alpha}{2}\left(e_{x}e_{y}\right)\right]$$

$$Bias\left(\widehat{\bar{Y}}_{prop}^{m}\right)=\bar{Y}\left[\frac{\alpha}{4}\vartheta\left(\frac{\lambda}{2-\lambda}\right)C_{x}^{2}+\frac{\alpha^{2}}{8}\vartheta\left(\frac{\lambda}{2-\lambda}\right)C_{x}^{2}-\frac{\alpha}{2}\vartheta\left(\frac{\lambda}{2-\lambda}\right)C_{yx}\right]$$

$$Bias\left(\widehat{\bar{Y}}_{prop}^{m}\right)=\left(\frac{\lambda}{2-\lambda}\right)\vartheta\bar{Y}\left[C_{x}^{2}\left(\frac{\alpha}{4}+\frac{\alpha^{2}}{8}\right)-\frac{\alpha}{2}C_{yx}\right] \tag{15}$$

Now squaring the equation (14) to optimize the MSE of a proposed estimator, we get

$$\left(\widehat{\bar{Y}}_{prop}^{m}-\bar{Y}\right)^{2}=\bar{Y}^{2}\left[e_{y}-\frac{\alpha e_{x}}{2}+\frac{\alpha e_{x}^{2}}{4}+\frac{\alpha^{2}e_{x}^{2}}{8}-\frac{\alpha e_{x}e_{y}}{2}\right]^{2}$$

$$\left(\widehat{\bar{Y}}_{prop}^{m}-\bar{Y}\right)^{2}=\bar{Y}^{2}\left[e_{y}^{2}+\frac{1}{4}\alpha^{2}e_{x}^{2}-\alpha e_{y}e_{x}\right]$$

Take the expectation on both the sides of above equation to derive the MSE of a proposed estimator up to the first order of approximation, we get

$$E\left(\widehat{\bar{Y}}_{prop}^{m}-\bar{Y}\right)^{2}=\bar{Y}^{2}\left[E\left(e_{y}^{2}\right)+\frac{\alpha^{2}}{4}E\left(e_{x}^{2}\right)-\alpha E\left(e_{y}e_{x}\right)\right]$$

$$MSE\left(\widehat{\bar{Y}}_{prop}^{m}\right)=\left(\frac{\lambda}{2-\lambda}\right)\bar{Y}^{2}\left[\vartheta C_{y}^{2}+\frac{1}{4}\alpha^{2}\theta\, C_{x}^{2}-\alpha\,\theta\,\rho C_{y}C_{x}\right]$$

$$MSE\left(\widehat{\bar{Y}}_{prop}^{m}\right)=\left(\frac{\lambda}{2-\lambda}\right)\vartheta\bar{Y}^{2}\left[C_{y}^{2}+\frac{1}{4}\alpha^{2}C_{x}^{2}-\alpha\rho C_{y}C_{x}\right]$$

Now to optimize the value of α take the derivation with respect to α of $MSE\left(\widehat{\bar{Y}}_{prop(SRS)}^{m}\right)$ we get

$$\frac{d}{d\alpha}MSE\left(\widehat{\bar{Y}}_{prop}^{m}\right)=0$$

$$\frac{d}{d\alpha}\left\{\left(\frac{\lambda}{2-\lambda}\right)\vartheta\bar{Y}^{2}\left[C_{y}^{2}+\frac{1}{4}\alpha^{2}C_{x}^{2}-\alpha\rho C_{y}C_{x}\right]\right\}=0$$

$$\vartheta\bar{Y}^{2}\left[0+\frac{1}{2}\alpha C_{x}^{2}-\rho C_{y}C_{x}\right]=0$$

$$\frac{1}{2}\alpha C_{x}^{2}=\rho C_{y}C_{x}$$

$$\alpha=\frac{2\rho C_{y}}{C_{x}}=\alpha^{*}$$

The minimum mean square error equation to the optimal values given by,

$$\boldsymbol{MSE\left(\widehat{\bar{Y}}_{prop_{min}}^{m}\right)=\left(\frac{\lambda}{2-\lambda}\right)\vartheta\bar{Y}^{2}\left[C_{y}^{2}+\frac{1}{4}\alpha^{*2}C_{x}^{2}-\alpha^{*}\rho C_{y}C_{x}\right]} \tag{16}$$

4. Theoretical comparison

The following observations will be made when we compare the suggested estimator $\widehat{\bar{Y}}_{prop}^{m}$ with the existing estimators covered in Sect. (2).

Observation 1: from equation (1) and (16)

$MSE\left(\widehat{\bar{Y}}_{prop_{min}}^{m}\right)<MSE\left(\widehat{\bar{Y}}_{0}\right)$, if and only if $MSE\left(\widehat{\bar{Y}}_{0}\right)-MSE\left(\widehat{\bar{Y}}_{prop_{min}}^{m}\right)>0$,

Or if $\vartheta\bar{Y}^{2}\left[C_{y}^{2}-\left(\frac{\lambda}{2-\lambda}\right)\left[C_{y}^{2}+\frac{1}{4}\alpha^{*2}C_{x}^{2}-\alpha^{*}\rho C_{y}C_{x}\right]\right]>0$.

Observation 2: from equation (2) and (16)

$MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) < MSE\ (\hat{Y}_{cr})$, if and only if $MSE\ (\hat{Y}_{cr}) - MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) > 0$,

Or if $\vartheta \bar{Y}^2 \left[\left(C_y^2 + C_x^2 - 2\rho C_y C_x\right) - \left(\frac{\lambda}{2-\lambda}\right)\left(C_y^2 + \frac{1}{4}\alpha^{*2}C_x^2 - \alpha^{*}\rho C_y C_x\right)\right] > 0.$

Observation 3: from equation (3) and (16)

$MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) < MSE\ (\hat{Y}_{Mu})$, if and only if $MSE\ (\hat{Y}_{Mu}) - MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) > 0$,

Or if $\vartheta \bar{Y}^2 \left[\left(C_y^2 + C_x^2 + 2\rho C_y C_x\right) - \left(\frac{\lambda}{2-\lambda}\right)\left(C_y^2 + \frac{1}{4}\alpha^{*2}C_x^2 - \alpha^{*}\rho C_y C_x\right)\right] > 0.$

Observation 4: from equation (4) and (16)

$MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) < MSE\ (\hat{Y}_{Pe})$, if and only if $MSE\ (\hat{Y}_{Pe}) - MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) > 0$,

Or if $\vartheta \bar{Y}^2 \left[\left(C_y^2 + \frac{1}{4}C_x^2 + \rho C_y C_x\right) - \left(\frac{\lambda}{2-\lambda}\right)\left(C_y^2 + \frac{1}{4}\alpha^{*2}C_x^2 - \alpha^{*}\rho C_y C_x\right)\right] > 0.$

Observation 5: from equation (5) and (16)

$MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) < MSE\ (\hat{Y}_{re})$, if and only if $MSE\ (\hat{Y}_{re}) - MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) > 0$,

Or if $\vartheta \bar{Y}^2 \left[\left(C_y^2 + \frac{1}{4}C_x^2 - \rho C_y C_x\right) - \left(\frac{\lambda}{2-\lambda}\right)\left(C_y^2 + \frac{1}{4}\alpha^{*2}C_x^2 - \alpha^{*}\rho C_y C_x\right)\right] > 0.$

Observation 6: from equation (6) and (16)

$MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) < MSE\ (\hat{Y}_{1Pr})$, if and only if $MSE\ (\hat{Y}_{1Pr}) - MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) > 0$,

Or if $\vartheta \left\{R_x^2 S_x^2 + S_y^2 (1-\rho^2) - \bar{Y}^2 \left(\frac{\lambda}{2-\lambda}\right)\left[C_y^2 + \frac{1}{4}\alpha^{*2}C_x^2 - \alpha^{*}\rho C_y C_x\right]\right\} > 0.$

Observation 7: from equation (7) and (16)

$MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) < MSE\ (\hat{Y}_{SD})$, if and only if $MSE\ (\hat{Y}_{SD}) - MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) > 0$,

Or if $\vartheta \left\{R_{SDx}^2 S_x^2 + S_y^2 (1-\rho^2) - \bar{Y}^2 \left(\frac{\lambda}{2-\lambda}\right)\left[C_y^2 + \frac{1}{4}\alpha^{*2}C_x^2 - \alpha^{*}\rho C_y C_x\right]\right\} > 0.$

Observation 8: from equation (8) and (16)

$MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) < MSE\ (\hat{Y}_{SK})$, if and only if $MSE\ (\hat{Y}_{SK}) - MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) > 0$,

Or if $\vartheta \left\{R_{SKx}^2 S_x^2 + S_y^2 (1-\rho^2) - \bar{Y}^2 \left(\frac{\lambda}{2-\lambda}\right)\left[C_y^2 + \frac{1}{4}\alpha^{*2}C_x^2 - \alpha^{*}\rho C_y C_x\right]\right\} > 0.$

Observation 9: from equation (9) and (16)

$MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) < MSE\ (\hat{Y}_{US1})$, if and only if $MSE\ (\hat{Y}_{US1}) - MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) > 0$,

Or if $\vartheta \left\{R_{US1x}^2 S_x^2 + S_y^2 (1-\rho^2) - \bar{Y}^2 \left(\frac{\lambda}{2-\lambda}\right)\left[C_y^2 + \frac{1}{4}\alpha^{*2}C_x^2 - \alpha^{*}\rho C_y C_x\right]\right\} > 0.$

Observation 10: from equation (10) and (16)

$MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) < MSE\ (\hat{Y}_{US2})$, if and only if $MSE\ (\hat{Y}_{US2}) - MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) > 0$,

Or if $\vartheta \left\{R_{US2x}^2 S_x^2 + S_y^2 (1-\rho^2) - \bar{Y}^2 \left(\frac{\lambda}{2-\lambda}\right)\left[C_y^2 + \frac{1}{4}\alpha^{*2}C_x^2 - \alpha^{*}\rho C_y C_x\right]\right\} > 0.$

Observation 11: from equation (11) and (16)

$$MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) < MSE\left(\widehat{\bar{Y}}^{m}_{Z}\right), \text{ if and only if } \quad MSE\left(\widehat{\bar{Y}}^{m}_{Z}\right) - MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) > 0,$$

Or if $\quad \vartheta\left(\frac{\lambda}{2-\lambda}\right)\bar{Y}^2\left\{\alpha^*\rho C_y C_x - \frac{1}{4}\alpha^{*2}C_x^2\right\} > 0.$

Observation 12: from equation (12) and (16)

$$MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) < MSE\left(\widehat{\bar{Y}}^{m}_{rmi}\right), \text{ if and only if } MSE\left(\widehat{\bar{Y}}^{m}_{rmi}\right) - MSE\left(\widehat{\bar{Y}}^{m}_{prop_{min}}\right) > 0,$$

Or if $\quad \left\{\left(\frac{\lambda}{2-\lambda}\right)\vartheta\bar{Y}^2\left[C_x^2 - \frac{1}{4}\alpha^{*2}C_x^2 + \alpha^*\rho C_y C_x - 2\rho C_y\, C_x\right]\right\} > 0.$

5. Numerical illustration

In this part, we are using data from (Kadilar and Cingi, 2003) for simple random sampling in this case, we have only considered data from the Aegean Region of Turkey. We used the data on apple production quantity (as the variate of interest) and number of apple trees (as the auxiliary variate) in 106 villages in the Aegean Region in 1999 (Source: Institute of Statistics, Republic of Turkey) to apply our suggested and other ratio estimators. The MSE of ratio estimators is calculated using these data, as explained in Table 2, and the MSE values of these estimators are compared to one another. In Table 3 PRE values of estimators also given.

The population data are displayed in Table 1. Considering the 86% correlation between the variables, we use a sample size of n = 20.

Table 1. Population

$N = 106$	$C_y = 5.22$	$S_y = 11551.53$	$R = 0.08$
$N = 20$	$C_y^2 = 27.25$	$S_y^2 = 133437845.34$	$R_{SD} = 0.08$
$\bar{Y} = 2212.59$	$C_x = 2.1$	$\rho = 0.86$	$R_{SK} = 0.08$
$\bar{Y}^2 = 4895554.51$	$C_x^2 = 4.41$	$\rho^2 = 0.74$	$R_{US1} = 0.002$
$\bar{X} = 27421.7$	$S_x = 57460$	$S_{yx} = 568176176.1$	$R_{US2} = 0.04$
$\bar{X}^2 = 751949630.89$	$S_x^2 = 3301651600$	$\alpha = 0.35$	$\alpha^2 = 0.12$
$\Theta = 0.04$	$\beta_{2(x)} = 34.57$		

Table 2. MSE Comparison of proposed estimator and existing estimators

$\hat{Y}_0$	5411348.28
$\hat{Y}_{cr}$	2542740.30
$\hat{Y}_{Mu}$	10031548.73
$\hat{Y}_{pe}$	7502499.45
$\hat{Y}_{re}$	3758095.23
$\hat{Y}_{1pr}$	2281806.68
$\hat{Y}_{SD}$	2281806.68
$\hat{Y}_{SK}$	2279646.31
$\hat{Y}_{US1}$	2256293.30
$\hat{Y}_{US2}$	2285052.26
$\widehat{\bar{Y}}^{m}_{Z_{\lambda=0.25}}$	773049.75
$\widehat{\bar{Y}}^{m}_{Z_{\lambda=0.50}}$	1803782.76
$\widehat{\bar{Y}}^{m}_{Z_{\lambda=0.75}}$	3246808.97
$\widehat{\bar{Y}}^{m}_{rmi_{\lambda=0.25}}$	363248.61
$\widehat{\bar{Y}}^{m}_{rmi_{\lambda=0.50}}$	847580.10
$\widehat{\bar{Y}}^{m}_{rmi_{\lambda=0.75}}$	1525644.18

$\hat{\bar{Y}}^{m}_{prop_{\lambda=0.25}}$	***201302.16***
$\hat{\bar{Y}}^{m}_{prop_{\lambda=0.50}}$	***469705.03***
$\hat{\bar{Y}}^{m}_{prop_{\lambda=0.75}}$	***845469.06***
$\hat{\bar{Y}}^{m}_{prop_{\lambda=1}}$	***1409115.09***

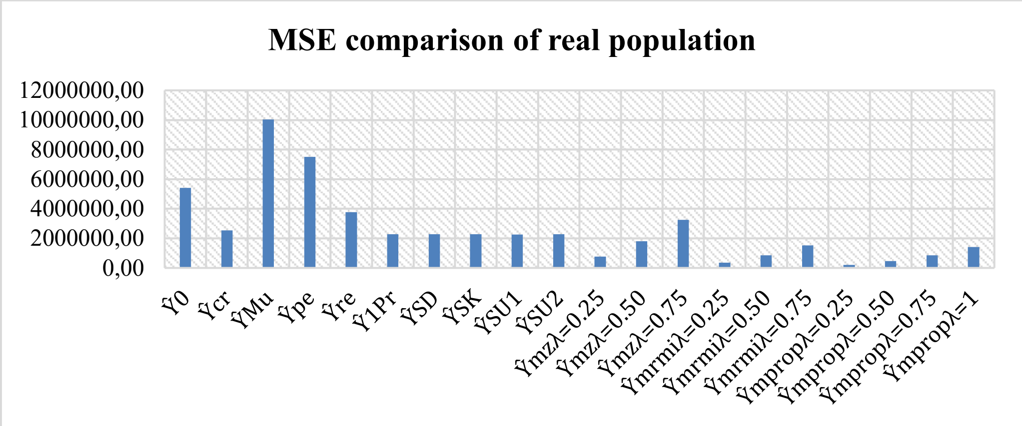

Figure 1. MSE Comparison of proposed estimator and existing estimators.

Here λ is representing the smoothing constant in proposed estimator which determines the weight assigned to different observations. The larger value of λ giving more weight to recent data while a smaller value of λ giving more weight to the past data. λ is having range from 0 to 1. We calculated the MSE for the different values of λ to check the efficiency of the proposed estimator. In Table 2 and Figure 1 the results are clearly indicates that the proposed estimator yields the lowest MSE for all values of λ, even at the stage we are considering $\lambda = 1$, where the proposed estimator reduces to standard form, it still maintains the lowest MSE compared to all existing estimators which is further validating the robustness and efficiency of the proposed estimator in population mean estimation.

Table 3. PRE Comparison of proposed estimator and existing estimators

$\hat{Y}_0$	*100.00*
$\hat{Y}_{cr}$	*212.82*
$\hat{Y}_{Mu}$	*53.94*
$\hat{Y}_{pe}$	*72.13*
$\hat{Y}_{re}$	*143.99*
$\hat{Y}_{1pr}$	*237.15*
$\hat{Y}_{SD}$	*237.15*
$\hat{Y}_{SK}$	*237.38*
$\hat{Y}_{US1}$	*239.83*
$\hat{Y}_{US2}$	*236.82*
$\hat{\bar{Y}}^{m}_{Z_{\lambda=0.25}}$	*700.00*
$\hat{\bar{Y}}^{m}_{Z_{\lambda=0.50}}$	*300.00*
$\hat{\bar{Y}}^{m}_{Z_{\lambda=0.75}}$	*166.68*
$\hat{\bar{Y}}^{m}_{rmi_{\lambda=0.25}}$	*1489.71*
$\hat{\bar{Y}}^{m}_{rmi_{\lambda=0.50}}$	*638.45*
$\hat{\bar{Y}}^{m}_{rmi_{\lambda=0.75}}$	*354.69*

$\hat{\bar{Y}}^{m}_{prop_{\lambda=0.25}}$	***2688.17***
$\hat{\bar{Y}}^{m}_{prop_{\lambda=0.50}}$	***1152.07***
$\hat{\bar{Y}}^{m}_{prop_{\lambda=0.75}}$	***640.04***
$\hat{\bar{Y}}^{m}_{prop_{\lambda=1}}$	***384.03***

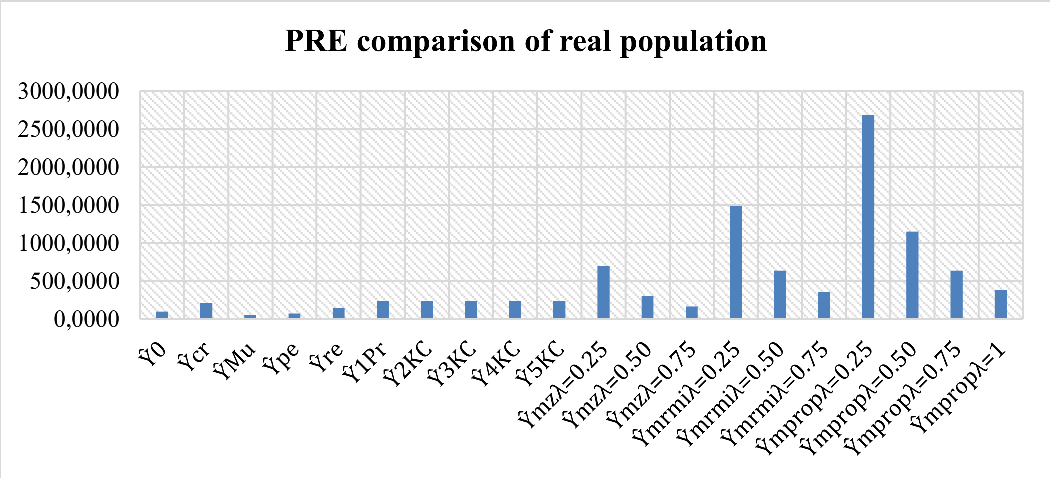

Figure 2. PRE Comparison of proposed estimator and existing estimators.

6. Simulated Data

In this section, we conduct a simulation study to support the theoretical results obtained for the proposed estimator under simple random sampling. A finite population of size N=1000 is generated from a bivariate normal distribution for the study variable Y and the auxiliary variable X using R studio. The population is generated with given means, variances, and correlation between Y and X, so that the structure of the data is like real survey situations. The main population characteristics used in the simulation are reported in Table 4 below.

Table 4. Simulated Population

$N = 106$	$C_y = 0.25$	$S_y = 9.2$	$R = 0.67$
$N = 20$	$C_y^2 = 0.06$	$S_y^2 = 84.62$	$R_{SD} = 0.67$
$\bar{Y} = 37.54$	$C_x = 0.2$	$\rho = 0.7$	$R_{SK} = 0.66$
$\bar{Y}^2 = 1409$	$C_x^2 = 0.05$	$\rho^2 = 0.48$	$R_{US1} = -1.23$
$\bar{X} = 56.41$	$S_x = 11.95$	$S_{yx} = 76.41$	$R_{US2} = 3.29$
$\bar{X}^2 = 3182.11$	$S_x^2 = 142.85$	$\alpha = 0.6$	$\alpha^2 = 0.36$
$\Theta = 0.04$	$\beta_{2(x)} = -0.54$		

Table 5. MSE comparison of simulated population

$\hat{Y}_0$	3.43
$\hat{Y}_{cr}$	1.87
$\hat{Y}_{Mu}$	10.12
$\hat{Y}_{pe}$	6.14
$\hat{Y}_{re}$	2.01
$\hat{Y}_{1pr}$	4.34
$\hat{Y}_{SD}$	4.34
$\hat{Y}_{SK}$	4.32
$\hat{Y}_{US1}$	4.38
$\hat{Y}_{US2}$	4.59

$\hat{\hat{Y}}^{m}_{Z_{\lambda=0.25}}$	*0.49*
$\hat{\hat{Y}}^{m}_{Z_{\lambda=0.50}}$	*1.14*
$\hat{\hat{Y}}^{m}_{Z_{\lambda=0.75}}$	*2.06*
$\hat{\hat{Y}}^{m}_{rmi_{\lambda=0.25}}$	*0.27*
$\hat{\hat{Y}}^{m}_{rmi_{\lambda=0.50}}$	*0.62*
$\hat{\hat{Y}}^{m}_{rmi_{\lambda=0.75}}$	*1.12*
$\hat{\hat{Y}}^{m}_{prop_{\lambda=0.25}}$	***0.25***
$\hat{\hat{Y}}^{m}_{prop_{\lambda=0.50}}$	***0.59***
$\hat{\hat{Y}}^{m}_{prop_{\lambda=0.75}}$	***1.06***
$\hat{\hat{Y}}^{m}_{prop_{\lambda=1}}$	***1.77***

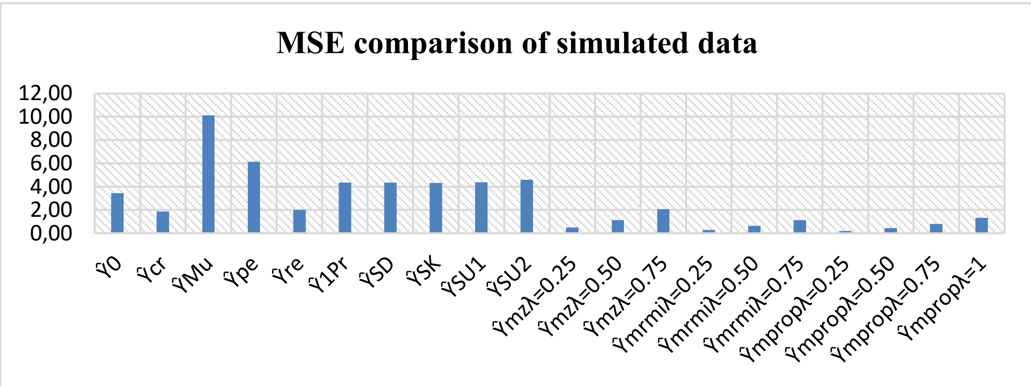

Figure 3. MSE comparison of simulated data.

Table 6. PRE comparison of simulated population

$\hat{Y}_{0}$	*100.00*
$\hat{Y}_{cr}$	*183.24*
$\hat{Y}_{Mu}$	*33.91*
$\hat{Y}_{pe}$	*55.94*
$\hat{Y}_{re}$	*170.65*
$\hat{Y}_{1pr}$	*79.09*
$\hat{Y}_{SD}$	*79.09*
$\hat{Y}_{SK}$	*79.44*
$\hat{Y}_{US1}$	*78.44*
$\hat{Y}_{US2}$	*74.77*
$\hat{\hat{Y}}^{m}_{Z_{\lambda=0.25}}$	*700.00*
$\hat{\hat{Y}}^{m}_{Z_{\lambda=0.50}}$	*300.00*
$\hat{\hat{Y}}^{m}_{Z_{\lambda=0.75}}$	*166.67*
$\hat{\hat{Y}}^{m}_{rmi_{\lambda=0.25}}$	*1282.69*
$\hat{\hat{Y}}^{m}_{rmi_{\lambda=0.50}}$	*549.73*
$\hat{\hat{Y}}^{m}_{rmi_{\lambda=0.75}}$	*305.40*
$\hat{\hat{Y}}^{m}_{prop_{\lambda=0.25}}$	***1354.03***
$\hat{\hat{Y}}^{m}_{prop_{\lambda=0.50}}$	***580.30***
$\hat{\hat{Y}}^{m}_{prop_{\lambda=0.75}}$	***322.39***
$\hat{\hat{Y}}^{m}_{prop_{\lambda=1}}$	***193.43***

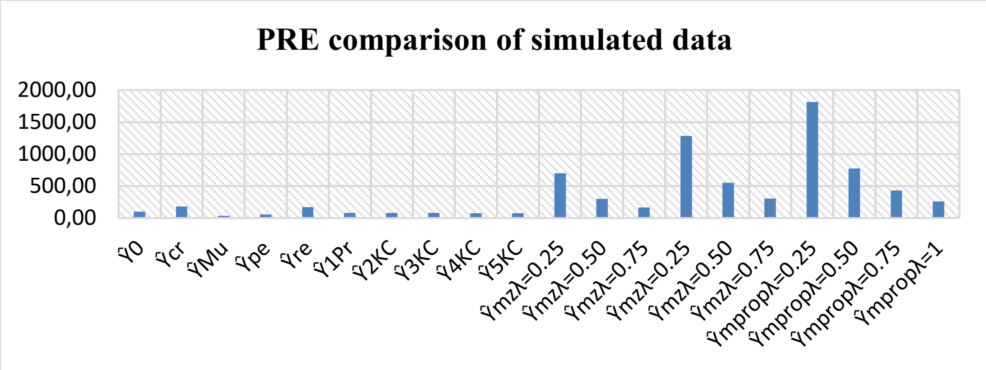

Figure 4. PRE comparison of simulated data.

7. Discussion

In this article we aim to proposed an estimator which has lowest mean square error to provide suitable inference. We worked on modified exponential ratio type estimator using simple random sampling. The mean square error of the proposed improved exponential-ratio estimator was evaluated using real and simulated data sets under simple random sampling. The result demonstrated that the estimator consistently shows lower MSE compared to existing estimators as shown in Table 2 for the real data and Table 5 for the simulated data set. Similarly, PRE shown in Table 3 for the real data and Table 6 for the simulated data set. Theoretically it is outperforming traditional methods, especially in simple random sampling from where it is efficiently leveraging auxiliary information. Overall, the proposed estimator provides more precise and efficient solution for population mean estimate which proving it superior to existing alternatives in certain conditions. Future research can be extended to more complex sampling designs like stratified, systematic and two stage sampling schemes in proposing the estimators using multi auxiliary variable or attributes.

Conflicts of Interest

The authors declare no conflict of interest.

Author Contributions

Conceptualization: JITENDRAKUMAR, B. R.; SAINI, M.; KUMAR, A. **Data curation:** JITENDRAKUMAR, B. R.; SAINI, M.; KUMAR, A. **Formal analysis:** JITENDRAKUMAR, B. R.; SAINI, M.; KUMAR, A. **Funding acquisition:** JITENDRAKUMAR, B. R.; SAINI, M.; KUMAR, A. **Investigation:** JITENDRAKUMAR, B. R.; SAINI, M.; KUMAR, A. **Methodology:** JITENDRAKUMAR, B. R.; SAINI, M.; KUMAR, A. **Project administration:** JITENDRAKUMAR, B. R.; SAINI, M.; KUMAR, A. **Software:** JITENDRAKUMAR, B. R.; SAINI, M.; KUMAR, A. **Resources:** JITENDRAKUMAR, B. R.; SAINI, M.; KUMAR, A. **Supervision:** JITENDRAKUMAR, B. R.; SAINI, M.; KUMAR, A. **Validation:** JITENDRAKUMAR, B. R.; SAINI, M.; KUMAR, A. **Visualization:** JITENDRAKUMAR, B. R.; SAINI, M.; KUMAR, A. **Writing - original draft:** JITENDRAKUMAR, B. R.; SAINI, M.; KUMAR, A. **Writing - review and editing:** JITENDRAKUMAR, B. R.; SAINI, M.; KUMAR, A.

References

1. Bahl, S., & Tuteja, R. Ratio and product type exponential estimators. *Journal of information and optimization sciences*, **12**(1), 159-164 (1991). https://doi.org/10.1080/02522667.1991.10699058
2. Cochran, W. G. The estimation of the yields of cereal experiments by sampling for the ratio of grain to total produce. *The journal of agricultural science*, **30**(2), 262-275 (1940).
3. Irfan, M., Javed, M., & Haider Bhatti, S. Difference-type-exponential estimators based on dual auxiliary information under simple random sampling. *Scientia Iranica*, **29**(1), 343-354 (2022). https://doi.org/10.24200/sci.2020.53592.3318
4. Kadilar, C., & Cingi, H. Ratio estimators in stratified random sampling. *Biometrical journal*, **45**(2), 218-225 (2003). https://doi.org/10.1002/BIMJ.200390007
5. Kadilar, C., & Cingi, H. Ratio estimators in simple random sampling. *Applied mathematics and computation*, **151**(3), 893-902 (2004). https://doi.org/10.1016/S0096-3003(03)00803-8
6. Kumar, A., Jitendrakumar, B. R., Singh Raghav, Y., & Saini, M. A new estimator for estimating population mean using two auxiliary attributes in stratified random sampling. *Lobachevskii Journal of Mathematics,* **44**(9), 3729-3739 (2023).
7. Murthy, M. N. Product method of estimation. *Sankhyā: The Indian Journal of Statistics*, Series A, 69-74 (1964).
8. Noor-ul-Amin, M. Memory type estimators of population mean using exponentially weighted moving averages for time scaled surveys. *Communications in Statistics - Theory and Methods,* **50**(12), 2747–2758 (2019). DOI: 10.1080/03610926.2019.1670850
9. Riyaz, S., Jan, R., Maqbool, S., Jan, T. R., & Rather, K. U. I. Generalized exponential ratio estimator of population mean using two auxiliary variables in simple random sampling with an application to agricultural data**.** *J. Stat. Appl. Probab. Lett*, **9**(2), 89-99 (2022). https://doi.org/10.18576/jsapl/090204
10. Roberts, S. W. Control Chart Tests Based on Geometric Moving Averages. *Technometrics*, **1**(3), 239–250 (1959).
11. Saini, M., Jitendrakumar, B. R., & Kumar, A. Optimum estimator in simple random sampling using two auxiliary attributes with application in agriculture, fisheries and education sectors. *MethodsX*, **9**, 101915 (2022). https://doi.org/10.1016/j.mex.2022.101915
12. Saini, M., Jitendrakumar, B. R., & Kumar, A. Improved ratio estimator under simple and stratified random sampling. *Life Cycle Reliability and Safety Engineering*, **13**(2), 181-187 (2024). https://doi.org/10.1007/s41872-024-00253-x
13. Shabbir, J., & Gupta, S. A. T. On improvement in variance estimation using auxiliary information. *Commun. Stat. Theory Methods* **36**(12), 2177–2185 (2007).
14. Singh, H. P., Tailor, R., Tailor, R., & Kakran, M. S. An improved estimator of population mean using power transformation. *Journal of the Indian Society of Agricultural Statistics*, **58**(2), 223-230 (2004).
15. Singh, H. P., Singh, S., & Kozak, M. A family of estimators of finite-population distribution function using auxiliary information. *Acta applicandae mathematicae*, **104**, 115-130 (2008). https://doi.org/10.1007/s10440-008-9243-1
16. Sisodia, B. V. S., & Dwivedi, V. K. A modified ratio estimator using coefficient of variation of auxiliary variable. *Journal of the Indian society of Agricultural Statistics*, **33**(1), 13-18 (1981).
17. Srivastava, S. K. An estimator using auxiliary information in sample serveys. *Calcutta Statistical Association Bulletin,* **16**(2-3), 121-132 (1967).
18. Yadav, S. K., Kadilar, C., Shabbir, J., & Gupta, S. Improved family of estimators of population

variance in simple random sampling. *Journal of Statistical Theory and Practice,* **9**, 219-226 (2015). https://doi.org/10.1080/15598608.2013.856359

19. Zaman, T., Dünder, E., Audu, A., Alilah, D. A., Shahzad, U., & Hanif, M. Robust regression-ratio-type estimators of the mean utilizing two auxiliary variables: a simulation study. *Mathematical Problems in Engineering*, **2021**(1), 6383927 (2021). https://doi.org/10.1155/2021/6383927?urlappend=%3Futm_source%3Dresearchgate.net%26utm_medium%3Darticle
20. Zaman, T., Bulut, H., & Yadav, S. K. Robust ratio-type estimators for finite population mean in simple random sampling: a simulation study. *Concurrency and Computation: Practice and Experience*, **34**(25), e7273 (2022). https://doi.org/10.1002/cpe.7273