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ARTICLE

# The reflected-shifted-truncated Maxwell distribution: properties and applications to reliability analysis

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**Abstract**

Modeling lifetime and reliability data requires flexible probability distributions capable of capturing diverse hazard rate behaviors and skewness patterns. Classical models often fail to adequately represent left-skewed data with bounded ranges. To address this limitation, the Reflected-Shifted-Truncated Maxwell (RSTM) distribution is introduced as an extension of the classical Maxwell model through reflection, shifting, and truncation. Key statistical properties including moments, hazard rate behavior, and stress–strength reliability are derived. Parameters are estimated using the maximum likelihood method for both complete and right-censored data, and estimator performance is assessed via simulation studies. The effectiveness of the RSTM distribution is illustrated through two fiberglass strength datasets, representative of left-skewed lifetime data. Comparative analysis based on information-theoretic measures demonstrates that the RSTM distribution consistently outperforms competing models, underscoring its potential as a robust tool for modeling left-skewed lifetime and reliability data.

**Keywords**: Maxwell Distribution, Negative Skewed data, Maximum Likelihood Estimation, Right Censoring.

## 1. Introduction

In statistical modeling, probability distributions are essential as they allow researchers to model the variability and patterns present in datasets. There are hundreds of distributions available, developed over the past decades, and they are extensively used in various fields such as engineering, actuarial science, environmental science, medical sciences, biology, demographic studies, economics, finance, and insurance.

In the analysis of lifetime data, researchers are particularly interested in developing new distributions to improve their ability to fit lifetime data. Parametric lifetime models, such as the exponential, Weibull, Lomax, Lindley, and Gamma distributions, are commonly used in reliability theory and survival analysis. These models help researchers to understand the distribution of lifetimes or time to failure in a population and make predictions about future events based on past data. Recently, Maxwell distribution has becomes a popular life-time model, it was firstly introduced in literature by Maxwell, J. C. and Clerk, J. (1860) and further, described by Boltzman, L. (1870) with few assumptions. The Maxwell distribution as a lifetime model has been used by Tyagi, R. K. and Bhattacharya, S. K. (1989), They also found the statistical properties like Minimum Variance Unbiased Estimator (MVUE), Reliability function and Bayes estimator of the parameter of this distribution. Moreover,

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Bekker, A. and Roux, J. J. J. (2005) also studied empirical Bayes estimation for the Maxwell distribution. The generalized Maxwell distribution that was used by Sharma, V.K., Singh, S. K. and Singh, U. (2014), evaluated the applicability of the distribution for Rainfall data and also compared with three parameter Weibull and Gamma distribution. These studies highlight the versatility and usefulness of the Maxwell distribution in various fields, including reliability theory and environmental sciences.

Tomer, S. K. and Panwar, M. S. (2015) obtained maximum likelihood and Bayes estimates of parameter under Type-I progressive hybrid censoring for Maxwell distribution. Therefore, maxwell distribution is very useful in life-testing and reliability models, the failure rate is not realistic and many distribution are constant lifetime model but the problem is that which distribution will be correct to solve the problem. To best fitted data, there is a lot of models are available with difference type of failure rates. Researchers often need to carefully select a model that best fit the data based on the characteristics of the dataset and the underlying assumptions of the models. However, in many real life problems, it has been seen that failure rate increasing or decreasing and some time noticed that the failure rate reaches to its peak and then again decreases consistently. It has extensively studied in reliability theory by many authors such as Chaturvedi, A. and Rani, U. (1998), Kumari, A., Kumar, K. and Kumar, I. (2023), Mohd, I. and Sharma, A. K. (2024), Shams, M. and Mirzaie, M. A. (2025), Mohieddine, A. and Dalia, Z. (2025) and so on.

Earlier research by Dey, S., Altun, E., Kumar, D. and Ghosh, I. (2023) and Dey, S., Waymyers, S. D. and Kumar, D. (2020) introduced the Reflected-Shifted-Truncated technique to model left-skewed lifetime data using Lindley and Lomax distributions, demonstrating greater flexibility in capturing increasing and peak-shaped hazard rates. Motivated by these researchers, we extend the Reflected-Shifted-Truncated methodology to Reflected-Shifted-Truncated Maxwell distribution in life time modeling and study its important properties. The main aim of taking this study is to introduce a new distribution i.e., RSTM distribution to model left skewed data by using Reflected-Shifted-Truncated methodology and discuss the uses, outline its features in detail, and suggest a few potential expansions. Classical lifetime models such as Exponential, Weibull, Lindley, Lomax, Gamma, and Maxwell distributions often fail to model left-skewed data and complex hazard shapes (increasing/decreasing/peak–shaped).

Since the Maxwell distribution has useful properties but lacks flexibility, we applied the Reflected-Shifted-Truncated (RST) method to create a more flexible version: the RSTM distribution. The RSTM distribution differs from the RST Gamma and RST Lindley models in terms of the parent distribution. While RST Gamma generally creates heavier tails and RST Lindley produces lighter tails, the RSTM distribution takes on the Maxwell distribution's moderate tail characteristics. The RSTM model can effectively handle left-skewness, bounded data, and different hazard-rate shapes, which classical models cannot achieve. This results in smooth left-skewness and balanced tail weight, making flexible in detecting left-skewed behavior and different hazard rate shapes and appropriate for bounded lifetime data.

The rest of article is arranged in such a manner that section 3 describes a range of mathematical properties of the new distribution i.e., RSTM distribution. Stress-Strength Reliability is introduced in section 4. Section 5 discusses estimation of the parameters using the method of maximum likelihood based on complete, variance-covariance matrix and right-censored data. In section 6, the simulation study is conducted to assess the performance of the maximum likelihood estimators. Finally, some concluding remarks are given in section 7.

## 2. RST Maxwell Distribution

Maxwell distribution has been proposed by Maxwell, J. C. and Clerk, J. (1860). Recently, Tomer, S. K. and Panwar, M. S. (2020) reviewed the Inverse Maxwell distribution and established its important statistical properties with its applications. A real valued random variable $Y$ follows Maxwell distribution with probability density function (*pdf*) as

$$f(y;\theta) = \frac{4y^2}{\sqrt{\pi}\theta^{\frac{3}{2}}} \exp\left\{-\frac{y^2}{\theta}\right\}; \quad y > 0, \theta > 0.$$

If we reflect the Maxwell-Bolzmann distribution about the $y$-axis and shift it $k$ units to the right, we get

$$f_1(y;\theta) = \frac{4(-y+k)^2}{\sqrt{\pi}\theta^{\frac{3}{2}}} \exp\left\{-\frac{(-y+k)^2}{\theta}\right\}; \quad 0 < y > k, \theta, k > 0.$$

The cumulative density function (c.d.f.) of this Reflecting-Shifted Maxwell distribution is

$$F_1(y|\theta, k) = \frac{2}{\sqrt{\pi}}\left[\gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right) - \gamma\left(\frac{3}{2}, \frac{(k-y)^2}{\theta}\right)\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\Gamma\left(\frac{3}{2}, \frac{(k-y)^2}{\theta}\right) - \Gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right)\right]$$

where, $\gamma(a, z) = \int_0^z u^{a-1} e^{-u}\, du$ and $\Gamma(a, z) = \int_z^\infty u^{a-1} e^{-u}\, du$ are lower and upper incomplete gamma functions, respectively.
Truncating the Reflected-Shifted Maxwell distribution at 0 effectively restricts the new distribution to the interval $[0, k)$. The PDF for Reflected-Shifted-Truncated Maxwell distribution (RSTM) is given by,

$$f(y; \theta, k) = \frac{1}{F_1(k) - F_1(0)} f_1(y; \theta, k)$$

$$f(y; \theta, k) = \frac{2(-y+k)^2}{\theta^{\frac{3}{2}}\gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right)} \exp\left\{-\frac{(-y+k)^2}{\theta}\right\} \quad 0 < y > k, \theta, k > 0. \tag{1}$$

The cumulative density function (c.d.f.) of RSTM distribution is

$$F(y; \theta, k) = 1 - \frac{\gamma\left(\frac{3}{2}, \frac{(-y+k)^2}{\theta}\right)}{\gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right)}$$

$$= 1 - \frac{1 - \Gamma\left(\frac{3}{2}, \frac{(-y+k)^2}{\theta}\right)}{1 - \Gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right)} \tag{2}$$

The Reliability function of random variable $Y$ is define as

$$R(y; \theta, k) = \frac{\gamma\left(\frac{3}{2}, \frac{(-y+k)^2}{\theta}\right)}{\gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right)} \tag{3}$$

The Hazard function of random variable $Y$ is define as

$$h(y; \theta) = \frac{2(-y+k)^2}{\theta^{\frac{3}{2}}} \frac{\exp\left\{-\frac{(-y+k)^2}{\theta}\right\}}{\gamma\left(\frac{3}{2}, \frac{(-y+k)^2}{\theta}\right)}. \tag{4}$$

Figure 1 shows the performance of our distribution also shifted from right to left skewed and if value of k is approximately equal to θ then distribution behave like symmetric distribution like in figure θ = 5 and k=4.

Figure 2 presents the reliability function of the RSTM distribution, which decreases gradually with time. This indicates that as time progresses, the probability of survival reduces. The speed of decline depends on the parameter values, showing that larger values of θ maintain higher reliability for longer duration.

Figure 3 displays the cumulative distribution function (CDF) of the RSTM distribution, which increases steadily from 0 to 1. This reflects the cumulative probability of the random variable and highlights how the skewness of the distribution influences the steepness of the curve. In cases where $k \approx \theta$, the CDF exhibits a nearly symmetric S-shape pattern.

Figure 4 illustrates the hazard function of the RSTM distribution, showing the instantaneous rate of failure at a given time. The hazard rate may increase, decrease, or take a bathtub-like shape depending on parameter choices, making the RSTM distribution flexible for modeling different types of lifetime behaviors.

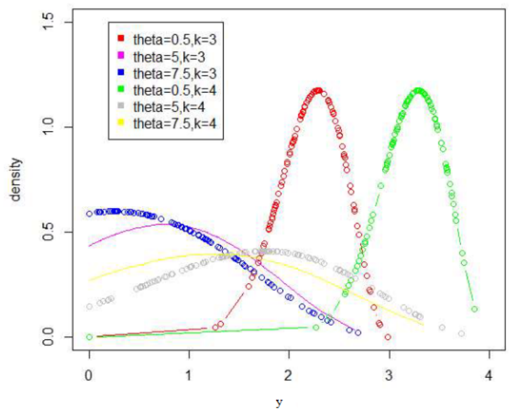


Figure 1. The probability density function of RSTM distribution ($\alpha$, $\theta$).

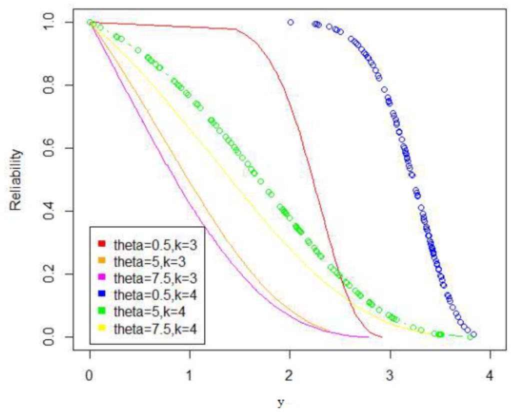


Figure 2. The Reliability function of RSTM distribution ($\alpha$, $\theta$).

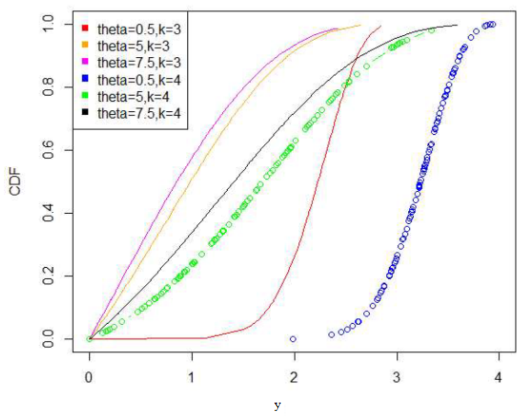


Figure 3. The CDF function of RSTM distribution ($\alpha$, $\theta$).

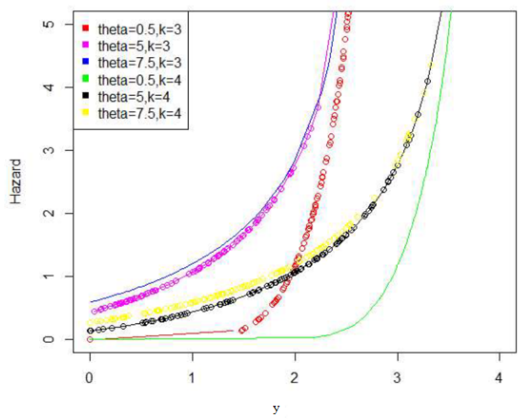


Figure 4. The Hazard function of RSTM distribution ($\alpha$, $\theta$).

## 2.1 Proposition 1

In this section, we discuss shape and characteristics of pdf $f(Y)$ and hazard rate function

**Theorem 1** *The RSTM is log-concave*

**Proof 1** *From eq. (1) we can write*

$$log(f(y|\theta, k)) = logC + 2log(-y+k) - \sum \left\{ -\frac{(-y+k)^2}{\theta} \right\}$$

$$where, C = \frac{2}{\theta^{\frac{3}{2}} \gamma \left( \frac{3}{2}, \frac{k^2}{\theta} \right)}$$

*Therefore,*

$$\frac{\partial^2 log(f(y|\theta,k))}{\partial^2 y} = -\left[\frac{2}{(-y+k)^2} + \frac{2}{\theta}\right]$$

*this implies that for* $\alpha > 0$, $k > 0$, *the second derivative is* $< 0$. *This completes the proof.*
*This implies that distribution is uni-model* $h(y)$ *is increasing in* $y$ *for all values of* $\theta$ *and* $k$. *(by Glaser, R. E. (1980) lemma).*

## 3. Mathematical Properties

In this section, we derive some important mathematical properties of the RSTM distribution, such as Moments, Quantile function, Measure of Skewness and Kurtosis, Mode, Stochastic Ordering and Stress-Strength Reliability.

### 3.1 Moments

The $r^{th}$ moment of a random variable $Y$ with probability density function from equation (1) is

$$\mu_r = E[Y^r] = \int_{-\infty}^{\infty} y^r f(y)dy = \int_0^k y^r \frac{2(-y+k)^2}{\theta^{\frac{3}{2}}\gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right)} \exp\left\{-\frac{(-y+k)^2}{\theta}\right\}$$

$$\mu_r = E[Y^r] = \frac{1}{\gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right)} \sum_{i=0}^{r} {}^rC_i(-1)^i\theta^{\frac{i}{2}}k^{r-i}\gamma\left(\frac{i+3}{2}, \frac{k^2}{\theta}\right) \tag{5}$$

### 3.2 Quantile function

The $p^{th}$ quantile $y_p$ of RSTM distribution can be obtained by solving the equation

$$p = F_Y(y_q, ; \theta, k)$$

The C.D.F. of the RSTM distribution defined in eq. (6) is

$$= 1 - \frac{1 - \Gamma\left(\frac{3}{2}, \frac{(-y+k)^2}{\theta}\right)}{1 - \Gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right)}$$

We know that, value of $k$, $\theta$ and $x_p$ is known quantity,
therefore,

$$p = 1 - \frac{1 - \Gamma\left(\frac{3}{2}, \frac{(-y_p+k)^2}{\theta}\right)}{1 - \Gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right)}$$

$$1 - p = \frac{1 - \Gamma\left(\frac{3}{2}, \frac{(-y_p+k)^2}{\theta}\right)}{1 - \Gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right)}$$

where, $P = 1 - \left[1 - \Gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right)\right](1-p)$

$$P = \Gamma\left(\frac{3}{2}, \frac{(-y_p+k)^2}{\theta}\right)$$

$$\frac{(-y_p+k)^2}{\theta} = G^{-1}\left(\frac{3}{2}, P\right)$$

$$y_p = k - \left(\theta G^{-1}\left(\frac{3}{2}, P\right)\right)^{\frac{1}{2}} \tag{6}$$

where $G^{-1}\left(\frac{3}{2}, P\right)$ is an inverse gamma regularized function and can be approximated by using following series.

$$G^{-1}(z,a) = (-(z-1)\Gamma(a+1))^{\frac{1}{a}} + \frac{\left[(-(z-1)\Gamma(a+1))^{\frac{1}{a}}\right]^2}{a+1} + \frac{(3a+5)\left[(-(z-1)\Gamma(a+1))^{\frac{1}{a}}\right]^3}{2(a+1)^2(a+2)} + \dots$$

## 3.3 Measure of Skewness and Kurtosis

To assess outliers or issue of robustness, the roles of skewness and kurtosis are well known. Here, on using the expression of quantile given in (2), we obtain the Galton's measure of skewness and Moor's measure of kurtosis (see Gilchrist, W. (2000)). These measures are given, respectively, by

$$G(\theta) = \frac{y_{\frac{3}{4}}(\theta) + y_{\frac{1}{4}}(\theta) - 2y_{\frac{1}{2}}(\theta)}{y_{\frac{3}{4}}(\theta) - y_{\frac{1}{4}}(\theta)}$$

and

$$K(\theta) = \frac{y_{\frac{7}{8}}(\theta) - y_{\frac{5}{8}}(\theta) + y_{\frac{3}{8}}(\theta) - y_{\frac{1}{8}}(\theta)}{y_{\frac{3}{4}}(\theta) - y_{\frac{1}{4}}(\theta)}$$

The range of Galton's measure of skewness $G(.)$ is $(-1, 1)$; for a perfectly symmetrical distribution $G(.) = 0$, A large and positive $G(.)$ indicates a long tail to the right, a positive skewness, and vice versa. Both the measures calculated above are alternative measures and are based on quantiles. Moor's measure of kurtosis is not influenced by the (extreme) tails of the distribution, and the calculation is simple.

**Table 1.** Mean, Median, Variance, Skewness and Kurtosis of the RSTM for varying values of θ and k

| k | θ | Mean | Median | Mode | Variance | Skewness | kurtosis |
|---|---|---|---|---|---|---|---|
| 3 | 0.5 | 2.2134 | 2.1980 | 2.2928 | 0.1011 | -0.4032 | 2.8118 |
| | 2.5 | 1.3087 | 1.3225 | 1.4189 | 0.3290 | 0.0750 | 2.3887 |
| | 5 | 0.9572 | 0.9802 | 0.7639 | 0.3790 | 0.2578 | 2.2571 |
| | 7.5 | 0.9201 | 0.9068 | 0.2613 | 0.3532 | 0.2541 | 2.1926 |
| 4 | 0.5 | 3.1829 | 3.2247 | 3.2928 | 0.1037 | -0.2575 | 2.4136 |
| | 2.5 | 2.1933 | 2.1679 | 2.4189 | 0.4591 | -0.0471 | 2.3536 |
| | 5 | 1.5116 | 1.4547 | 1.7639 | 0.7054 | 0.4140 | 2.1943 |
| | 7.5 | 1.4765 | 1.3909 | 1.2614 | 0.7244 | 0.2994 | 2.0975 |
| 5 | 0.5 | 4.1670 | 4.1497 | 4.2928 | 0.1235 | -0.1586 | 2.3584 |
| | 2.5 | 3.3768 | 3.4675 | 3.4189 | 0.4158 | -0.1639 | 2.1618 |
| | 5 | 2.6257 | 2.6992 | 2.7639 | 1.0125 | -0.4828 | 2.1264 |
| | 7.5 | 2.1738 | 2.2468 | 2.2613 | 1.5028 | 0.0256 | 2.0753 |
| 10 | 0.5 | 9.2219 | 9.2168 | 9.2928 | 0.1277 | -0.4840 | 2.7165 |
| | 2.5 | 8.1517 | 8.2389 | 8.4188 | 0.5976 | -0.3034 | 2.6369 |
| | 5 | 7.5295 | 7.6201 | 7.7639 | 1.2368 | -0.4505 | 2.5295 |
| | 7.5 | 7.0411 | 7.0236 | 7.2613 | 1.4266 | -0.2977 | 2.2643 |

Table 1 illustrates the mean, median, variance, skewed and kurtosis of the RSTM distribution for different values of θ and $k$. For increase in the value of θ and when k is fixed, the mean, median, mode, skewness, and kurtosis are decreases whereas variance increases. Similarly, for varying values of k when θ is fixed then mean, median, mode, variance increases whereas skewness and kurtosis decreases, which indicates that when parameters are varies the behaviour of our model also varies from right skewed to left skewed.

## 3.4 Mode

The mode of $f(y)$ is the root of the equation

$$f(y;\theta,k) = \frac{2(-y+k)^2}{\theta^{\frac{3}{2}}\gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right)} \exp\left\{-\frac{(-y+k)^2}{\theta}\right\}$$

for finding the mode, we take

$$\frac{\partial log(f(y|\theta,k))}{\partial y}=0$$

$$\frac{-2}{(-y+k)}+\frac{2(-y+k)}{\theta}=0$$

$$y_0=k-\sqrt{\theta}$$

## 3.5 Stochastic Ordering

A random variable is said to be stochastically greater than $Y$ or $(Y<X)$ if $F_Y(t)\leq F_X(t)$ for all $t$ In the similar way, $X$ is said to be greater than $Y$ if,

i Hazard rate order $(Y\leq_{hr}X)$ if $h_Y(t)\geq h_X(t)$ for all $t$.
ii Mean residual life order $(Y\leq_{ml}X)$ if $m_Y(t)\geq m_X(t)$ for all $t$.
iii Likelihood ratio order $(Y\leq_{lr}X)$ if $\frac{f_X(t)}{f_Y(t)}$ decreases in $t$.

**Theorem 2** *Let $X\sim RSTM(\theta_1,k_1)$ and $Y\sim RSTM(\theta_2,k_2)$ if $X<min(k_1,k_2)$ then $X\geq_{lr}Y$, $X\geq_{mlr}Y$, $X\geq_{hr}Y$ and $X\geq_{st}Y$.*

**Proof 2** *The likelihood ratio is*

$$\frac{f_X(x)}{f_Y(x)}=\frac{\frac{2(-x+k_1)^2}{\theta_1^{\frac{3}{2}}\gamma\left(\frac{3}{2},\frac{k_1^2}{\theta_1}\right)}\exp\left\{-\frac{(-x+k_1)^2}{\theta_1}\right\}}{\frac{2(-x+k_2)^2}{\theta_2^{\frac{3}{2}}\gamma\left(\frac{3}{2},\frac{k_2^2}{\theta_2}\right)}\exp\left\{-\frac{(-x+k_2)^2}{\theta_2}\right\}}$$

*Thus,*

$$\frac{\partial}{\partial x}log\frac{f_X(x)}{f_Y(x)}=\left(\frac{2}{(-x+k_2)}-\frac{2}{(-x+k_1)}\right)-\frac{2(-x+k_2)}{\theta_2}-\frac{2(-x+k_1)}{\theta_1}$$

*Now, if, $X<min(k_1,k_2)$ then $\frac{\partial}{\partial x}log\frac{f_X(x)}{f_Y(x)}\geq 0$. This implies that $Y$ is stochastically smaller than $X$ with respect to likelihood ratio i.e., $X\geq_{lr}Y$ and similarly we can conclude for $X\geq_{mlr}Y$, $X\geq_{hr}Y$ and $X\geq_{st}Y$.*
*Regarding these orderings, we got the results from Shaked, M. and Kumar, S. and George, J. (2007) who conveys that the existence of likelihood ratio ordering implies the existence of all orderings mentioned above. Here, we claim that RSTM rv's are ordered with respect to the strongest ordering, that is, the likelihood ratio ordering.*

# 4. Stress-Strength Reliability

Let $X$ be the strength of a system under stress $Y$, and if $X$ follows RSTM$(\theta_1,k_1)$ and $Y$ follows RSTM$(\theta_2,k_2)$, $X$ and $Y$ are statistically independent random variables, then $R=P(Y<X)$, the measure of system performance.

Stress-Strength Reliability Metrics is provided by

$$R=P(Y<X)=\int_0^\infty f_1(x)F_2(x)dx$$

$$=\int_0^\infty\left[\frac{2(-y+k_1)^2}{\theta_1^{\frac{3}{2}}\gamma\left(\frac{3}{2},\frac{k_1^2}{\theta_1}\right)}\exp\left\{-\frac{(-y+k_1)^2}{\theta_1}\right\}\right]\left[1-\frac{\gamma\left(\frac{3}{2},\frac{(-y+k_2)^2}{\theta_2}\right)}{\gamma\left(\frac{3}{2},\frac{k_2^2}{\theta_2}\right)}\right]dy$$

The equation are in implicit form, so it can be solved using numerical iteration, such as the Newton–Raphson method via Matlab or R software.

# 5. Maximum Likelihood (ML) Estimation

## 5.1 ML Estimation for Complete Data

Let $y_1, y_2, ..., y_n$ be a random sample of size $n$ with probability density function given by (1). The likelihood function based on the observed sample is given by

$$L(x) = \left(\frac{2}{\theta^{\frac{3}{2}}}\right)^n \left[\gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right)\right]^n \prod_{i=0}^{n} (-y+k)^2 \exp\left\{-\frac{(-y+k)^2}{\theta}\right\}$$

then log-likelihood function is

$$lnL(\Theta) = nlog2 - \frac{3n}{2}log\theta - nlog\gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right) + \sum log(-y+k)^2 - \sum \frac{(-y+k)^2}{\theta} \quad (7)$$

Differentiate (7) w. r. to $\theta$, we have

$$\frac{\partial lnL(\Theta)}{\partial\theta} = \frac{3n}{2\theta} - n\frac{\gamma'\left(\frac{3}{2}, \frac{k^2}{\theta}\right)}{\gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right)} + \sum \frac{(-y+k)^2}{\theta^2} \quad (8)$$

Now, by leibniz integral rule Weisstein, E. W. (2003),

$$\frac{\gamma'\left(\frac{3}{2}, \frac{k^2}{\theta}\right)}{\gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right)} = \frac{n\frac{k^2}{\theta}^{\frac{3}{2}} - 1e^{-\frac{k^2}{\theta}}\frac{k^2}{\theta^2}}{\gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right)}$$

and Differentiate (7) w. r. to $k$, we have

$$\frac{\partial lnL(\Theta)}{\partial k} = \frac{n\gamma'\left(\frac{3}{2}, \frac{k^2}{\theta}\right)}{\gamma\left(\frac{3}{2}, \frac{k^2}{\theta}\right)} + \frac{2}{(-y+k)} - 2\frac{(-y+k)}{\theta} \quad (9)$$

Also, by Leibniz integral rule,

$$\gamma'\left(\frac{3}{2}, \frac{k^2}{\theta}\right) = e^{\frac{-k^2}{\theta}}\left(\frac{k^2}{\theta}\right)^{\frac{3}{2}}\frac{2k}{\theta}$$

The equations given by (8) and (9) can be solved simultaneously to find the maximum likelihood estimates of $\theta$ and k.

### 5.1.1 *Variance-Covariance Matrix*

We have obtained the variance-covariance matrix to find out the asymptotic confidence intervals. The observed information matrix for the parameter is given by inverting the matrix. Therefore, we get the observed approximate Fisher's Information matrix which is given by

$$I(\hat{\zeta}) = \begin{bmatrix} I_{\theta\theta} & I_{\theta k} \\ I_{kk} & I_{k\theta} \end{bmatrix}_{(\hat{\theta}, \hat{k})},$$

where, $\underline{\zeta} = (\theta, k)$ is the parameter vector, and

$$I_{\theta\theta} = -\frac{\partial^2 l}{\partial\theta^2} = -\frac{3n}{2\theta^2} + n\frac{\left[\gamma\left(\frac{3}{2},\frac{k^2}{\theta}\right)\gamma''\left(\frac{3}{2},\frac{k^2}{\theta}\right) - \gamma'\left(\frac{3}{2},\frac{k^2}{\theta}\right)^2\right]}{\left(\gamma\left(\frac{3}{2},\frac{k^2}{\theta}\right)\right)^2} + 2\sum_{i=1}^{n}\frac{(k-x_i)^2}{\theta^3}$$

$$I_{\theta k} = -\frac{\partial^2 l}{\partial\theta\partial k} = n\frac{\gamma\left(\frac{3}{2},\frac{k^2}{\theta}\right)\gamma''\left(\frac{3}{2},\frac{k^2}{\theta}\right) - \gamma'\left(\frac{3}{2},\frac{k^2}{\theta}\right)^2}{\left(\gamma\left(\frac{3}{2},\frac{k^2}{\theta}\right)\right)^2} - 2\sum_{i=1}^{n}\frac{(k-x_i)}{\theta^2}$$

$$I_{k\theta} = -\frac{\partial^2 l}{\partial k\partial\theta} = n\frac{\gamma\left(\frac{3}{2},\frac{k^2}{\theta}\right)\gamma''\left(\frac{3}{2},\frac{k^2}{\theta}\right) - \gamma'\left(\frac{3}{2},\frac{k^2}{\theta}\right)^2}{\left(\gamma\left(\frac{3}{2},\frac{k^2}{\theta}\right)\right)^2} - 2\sum_{i=1}^{n}\frac{(k-x_i)}{\theta^2}$$

$$I_{kk} = -\frac{\partial^2 l}{\partial k^2} = n\frac{\gamma\left(\frac{3}{2},\frac{k^2}{\theta}\right)\gamma''\left(\frac{3}{2},\frac{k^2}{\theta}\right) - \gamma'\left(\frac{3}{2},\frac{k^2}{\theta}\right)^2}{\left(\gamma\left(\frac{3}{2},\frac{k^2}{\theta}\right)\right)^2} + 2\sum_{i=1}^{n}\frac{1}{(k-x_i)^2} + \frac{2}{\theta}$$

by Leibniz integral rule,

$$\gamma'\left(\frac{3}{2},\frac{k^2}{\theta}\right) = e^{\frac{-k^2}{\theta}}\left(\frac{k^2}{\theta}\right)^{\frac{3}{2}}\frac{2k}{\theta}$$

Similarly, the variance-covariance matrix of $\theta$ and $k$ can be solved simultaneously

## 5.2 ML estimation for Right Censored Data

Let $y_1, y_2, ..., y_n$ be a right censored random sample of size with probability density function given by equation (1). The censoring indicator with probability density function is also given in equation (10) and $\delta_i$ is the censoring indicator such that $\delta_i$ is 1 is observed, and $\delta_i$ is 0 if the event of interest is not observed (event time is right-censored). The likelihood function is given by

$$L(\Theta) = \prod_{i=1}^{n}\left[\frac{\frac{2}{\theta^{\frac{3}{2}}}(-y+k)^2 e^{-\frac{(-y+k)^2}{\theta}}}{\gamma\left(\frac{3}{2},\frac{k^2}{\theta}\right)}\right]^{\delta_i}\left[\frac{\gamma\left(\frac{3}{2},\frac{(-y+k)^2}{\theta}\right)}{\gamma\left(\frac{3}{2},\frac{k^2}{\theta}\right)}\right]^{1-\delta_i} \tag{10}$$

where $\theta > 0$, $k > 0$ and $0 < y_i < k$.
The corresponding log-likelihood function is given by

$$l(\Theta) = logL(\Theta)$$

$$l(\Theta) = \sum_{i=1}^{n}\delta_i\left[log2 - \frac{3}{2}log\theta + 2log(-y+k) - \frac{(-y+k)^2}{\theta}\right]$$
$$+(1-\delta_i)\left[\gamma\left(\frac{3}{2},\frac{(-y+k)^2}{\theta}\right) - \gamma\left(\frac{3}{2},\frac{k^2}{\theta}\right)\right] \tag{11}$$

The maximum likelihood estimators of $\Theta$ are the simultaneous solutions of the equations

$$\frac{\partial l(\Theta)}{\partial\theta} = \sum_{i=1}^{n}\delta_i\left[-\frac{3}{2\theta} + \frac{(-y+k)^2}{\theta^2}\right] + (1-\delta_i)\left[\frac{\gamma'\left(\frac{3}{2},\frac{(-y+k)^2}{\theta}\right)}{\gamma\left(\frac{3}{2},\frac{(-y+k)^2}{\theta}\right)} - \frac{\gamma'\left(\frac{3}{2},\frac{k^2}{\theta}\right)}{\gamma\left(\frac{3}{2},\frac{k^2}{\theta}\right)}\right] \tag{12}$$

Therefore, by lebiniz integral rule,

$$\gamma'\left(\frac{3}{2},\frac{k^2}{\theta}\right) = \left(\frac{k^2}{\theta^2}\right)\left(\frac{k^2}{\theta}\right)^{\frac{1}{2}} e^{-\frac{k^2}{\theta}}$$

Similarly

$$\gamma'\left(\frac{3}{2},\frac{(-y+k)^2}{\theta}\right) = \left(\frac{(-y+k)^2}{\theta^2}\right)\left(\frac{(-y+k)^2}{\theta}\right)^{\frac{1}{2}} e^{-\frac{(-y+k)^2}{\theta}}$$

$$\frac{\partial l(\Theta)}{\partial k} = \sum_{i=1}^{n} \delta_i \left[ -\frac{2}{(-\gamma + k)} + \frac{2(-\gamma + k)}{\theta} \right] + (1 - \delta_i) \left[ \frac{\gamma' \left( \frac{3}{2}, \frac{(-\gamma+k)^2}{\theta} \right)}{\gamma \left( \frac{3}{2}, \frac{(-\gamma+k)^2}{\theta} \right)} - \frac{\gamma' \left( \frac{3}{2}, \frac{k^2}{\theta} \right)}{\gamma \left( \frac{3}{2}, \frac{k^2}{\theta} \right)} \right] \tag{13}$$

Now, by lebiniz integral rule,

$$\gamma' \left( \frac{3}{2}, \frac{k^2}{\theta} \right) = \left( \frac{2k}{\theta} \right) \left( \frac{k^2}{\theta} \right)^{\frac{1}{2}} e^{-\frac{k^2}{\theta}}$$

and similarly,

$$\gamma' \left[ \frac{3}{2}, \frac{(-\gamma + k)^2}{\theta} \right] = \left[ \left[ \frac{2(-\gamma + k)}{\theta} \right] \left[ \frac{(-\gamma + k)^2}{\theta} \right]^{\frac{1}{2}} e^{-\frac{(-\gamma+k)^2}{\theta}} \right]$$

The equations given by (12) and (13) can be solved simultaneously to find the maximum likelihood estimates of θ and k.

## 6. Simulation Study

To examine the behavior of the maximum likelihood estimators (MLEs) of the RSTM distribution for finite sample sizes, a Monte Carlo simulation study with 1000 replications was performed. The random samples from the RSTM$(\theta, k)$ distribution were generated using the quantile (inverse CDF) method. The $p^{th}$ quantile $y_p$ of the RSTM distribution is obtained as a solution of the (6) where $p \in (0, 1]$, We use this equation to generate the random numbers form RSTM$(\theta, k)$ distribution. To investigate the behavior of the ML estimators for finite sample of size $n$, we perform a simulation study caring out the following steps.

1 Choose the initial values $\theta_0$ and $k_0$ of the parameter vector $\Theta = (\theta, k)$ to define the RSTM distribution.
2 Choose a sample size $n$.
3 Generate $N$ independent samples of size $n$ from the RSTM$(\theta, k)$ distribution by using the quantile function in equation (6).
4 Compute the MLE $\Theta_n$ for each of the $N$ samples.
5 Compute the mean of the estimators over the $N$ samples and obtain the MSE as well as the average length and coverage probability of confidence intervals.

The simulated data is generated by using the necessary step described above and using the initial value of the parameter $\theta = 1.5, 2.50$ and $k = 5, 10$. The simulation study is done for different sample size $n = 10, 30, 50$, and 100 for 1000 iterations. Here we obtained the point and interval estimates for the unknown parameters. We present the estimated values of parameters along with mean square error (MSE) and interval estimates of parameters under asymptotic normality assumption with coverage probability.

**Table 2.** Estimated MSEs of MLE, average length of Confidence Interval with coverage probability of the parameters based on 1000 simulated data from the RSTM$(\theta, k)$ distribution with different values of $n$

| n | θ | k | $\hat{\theta}(MSE)$ | $CP_{\hat{\theta}}$ | $AL_{\hat{\theta}}$ | $\hat{k}(MSE)$ | $CP_{\hat{k}}$ | $AL_{\hat{k}}$ |
|---|---|---|---|---|---|---|---|---|
| 10 | 1.5 | 5 | 1.09(0.1302) | 0.87 | 2.7336 | 4.92(0.1510) | 0.83 | 1.1533 |
| | 2.50 | 5 | 1.17(0.1861) | 0.89 | 3.0670 | 4.77(0.1380) | 0.84 | 1.2575 |
| | 2.50 | 10 | 2.04(0.1302) | 0.88 | 4.3665 | 9.85(0.1801) | 0.81 | 1.4475 |
| 30 | 1.5 | 5 | 1.16(0.0993) | 0.93 | 1.0283 | 4.95(0.1070) | 0.88 | 0.5202 |
| | 2.50 | 5 | 2.40(0.0965) | 0.93 | 2.4670 | 4.93(0.0926) | 0.91 | 0.7252 |
| | 2.50 | 10 | 2.40(0.0798) | 0.87 | 2.6392 | 9.94(0.0924) | 0.88 | 0.8449 |
| 50 | 1.5 | 5 | 1.18(0.0412) | 0.90 | 0.7288 | 4.97(0.0540) | 0.93 | 0.3580 |
| | 2.50 | 5 | 1.24(0.0782) | 0.94 | 2.0592 | 4.97(0.0556) | 0.95 | 0.6504 |
| | 2.50 | 10 | 2.46(0.0320) | 0.89 | 1.4527 | 9.96(0.0347) | 0.94 | 0.5560 |
| 100 | 1.5 | 5 | 1.22(0.0215) | 0.94 | 0.5331 | 4.98(0.0135) | 0.83 | 0.2513 |
| | 2.50 | 5 | 2.46(0.0010) | 0.99 | 1.3773 | 4.98(0.0008) | 0.97 | 0.4353 |
| | 2.50 | 10 | 2.44(0.0018) | 0.92 | 1.0869 | 9.97(0.0021) | 0.94 | 0.3335 |

The Mean Squared Error (MSE) of parameter estimates, Coverage Probability (C.P.) and Average Length (A.L.) of confidence interval are represented in Table 2. we see that A.L. and MSE of estimator decreases as the sample size increases, which demonstrates that the parameters are consistent. Also, the estimated confidence internal will be obtained by the method of MLE using R software.

# 7. Illustrative Example: An application to Strength fiber glass data

The following glass fiber data are experimental strength values of two lengths, 1.5 cm and 15 cm, from the National Physical Laboratory in England (Smith, R. L. and Naylor, J. C. (1987)). Preliminary inspection of the data reveals possible outliers in the lower end point of the sample, the smallest observation in data-set 1 (strength value = 0.55) and the smallest two in data-set 2 (strength values = 0.37, 0.40). The fiberglass strength data exhibit left-skewness and bounded support, which are not adequately captured by classical lifetime models. The RSTM distribution, through its reflected–shifted–truncated structure, naturally accommodates this left-skewed behavior and allows for more appropriate tail and hazard-rate modeling, resulting in improved goodness of fit. This explanation has been added to enhance the practical interpretation of the results. The authors used RSTM distribution to model the two data sets and concluded that techniques appear to be better choice for handling unusually shaped likelihoods than the maximum likelihood techniques. We compare the RSTM performance to that of the other distributions. The RST Gamma distribution is not considered for the fiberglass datasets because it produces heavier tails, making it unsuitable for modeling the observed left-skewed data. Table 3 and 4 represents the estimated parameters with corresponding standard error for strength glass datasets.

**Table 3.** The estimated parameters with corresponding standard error for strength fiber glass data-set 1

| Model | Parameters | MLE(Se) | NLC | AIC | BIC |
|---|---|---|---|---|---|
| RST Lomax | $\theta$ | 30.9918(0.0625) | 39.5203 | 85.0406 | 95.6131 |
| | $\beta$ | 29.9805(0.08792) | | | |
| Lomax | $\theta$ | $4.0132 * 10^8$(0.0231) | 88.8303 | 181.6606 | 194.2331 |
| | $\Lambda$ | $6.0473 * 10^8$(0.1250) | | | |
| | $k$ | 2.2427(0.0527) | | | |
| Lindley | $\theta$ | 0.9961(0.0948) | 81.2284 | 164.4568 | 170.7431 |
| RST Lindley | $\theta$ | 1.5476(0.0231) | 36.9294 | 77.8588 | 90.4313 |
| | $k$ | 2.2420(0.6331) | | | |
| Maxwell | $\theta$ | 1.5826(0.0858) | 38.5913 | 79.0226 | 85.0387 |
| Inv. Maxwell | $\theta$ | 0.3723(0.1385) | 45.8830 | 93.7660 | 100.5230 |
| RST Maxwell | $\theta$ | 0.4980(0.0782) | 24.0670 | 54.1340 | 64.7065 |
| | $k$ | 2.3089(0.08792) | | | |

**Table 4.** The estimated parameters with corresponding standard error for strength fiber glass data-set 2

| Model | Parameters | MLE(Se) | NLC | AIC | BIC |
|---|---|---|---|---|---|
| Lomax | $\theta$ | $1.4984 * 10^8$(0.0301) | 51.6210 | 107.2420 | 119.8145 |
| | $\Lambda$ | $1.6933 * 10^8$(0.0149) | | | |
| RST Lomax | $\theta$ | 8.4185(0.0245) | 11.9074 | 29.8148 | 48.6736 |
| | $\beta$ | 9.3287(0.0722) | | | |
| | $k$ | 1.610(0.0753) | | | |
| Lindley | $\theta$ | 1.2741(0.0488) | 47.3559 | 96.7118 | 102.9987 |
| RST Lindley | $\theta$ | 2.3101(0.0138) | 9.2917 | 22.5834 | 35.1559 |
| | $k$ | 1.610(0.0613) | | | |
| Maxwell | $\theta$ | 0.8992(0.0098) | 16.4089 | 34.8178 | 41.1040 |
| Inv. Maxwell | $\theta$ | 0.7299(0.0498) | 25.4993 | 52.9986 | 59.2849 |
| RST Maxwell | $\theta$ | 0.2943(0.0068) | 8.0015 | 20.0030 | 32.5755 |
| | $k$ | 1.7375(0.0722) | | | |

From both tables, the NLC (Negative Likelihood Criteria), AIC (Akaike Information Criteria), and BIC (Bayesian Information Criteria) of RSTM distribution is lower than the other six models which indicates that the RSTM distribution provide the better fit among considered popular models.

# 8. Conclusion

In this section, we examine the findings from all tables and figures, demonstrating that the RSTM distribution is a part of the family of increasing failure rate distributions, and that its density function can shift from right-skewed to left-skewed as parameter values increase. The suggested two-parameter RSTM model is more flexible than classical lifetime distributions since it allows for limited support and a variety of hazard rate shapes.

Various significant statistical properties of the RSTM distribution were obtained, including moments, quantile function, skewness and kurtosis measurements, stochastic ordering, and stress-strength reliability. Maximum likelihood estimation was established for both complete and right-censored data, and a simulation study validated the estimators' consistency and finite-sample performance. Applications on real fiberglass strength datasets indicated that the RSTM distribution produces a better fit than competing models, as seen by decreased AIC, AICC, and BIC. Overall, the RSTM distribution is a versatile and efficient model for assessing left-skewed survival and reliability data.

## Conflict of Interest

All authors have declare no conflict of interest.

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