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ARTICLE

The new transformed Sine G family of distribution with inference and application.

Abdulhameed Ado Osi,[1] Yasphal Singh Raghav,[2] Shamsuddeen Ahmad Sabo[1,*], and Ibrahim Zakariyya Musa[1]

[1]Department of Statistics, Aliko Dangote University of Science and Technology Wudil, Kano, Nigeria
[2]Department of Mathematics, College of Science, Jazan University,Jazan, Kingdom of Saudi Arabia
[*]Corresponding author. Email: shamsuddeenahmadsabo@gmail.com

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Abstract
This article introduces the New Transformed Sine G Family of distributions, which is a new probability distribution based on trigonometric transformation. It provides detailed derivations of the statistical properties associated with this distribution, such as the hazard function, survival function, inverse hazard, quantile function, moments, moment generating function, and the median. The parameter estimation of the model was conducted using the maximum likelihood method, and the performance of the estimation method was evaluated through Monte Carlo simulation. Furthermore, the applicability of the proposed distribution was demonstrated by analyzing two real datasets, where the model showed superior fit compared to existing distributions for these datasets.
Keywords: Statistical inference, Maximum likelihood, Estimation, NTS-G Family Distribution.

1. Introduction

Statistical distributions are essential for understanding and analyzing various phenomena in the real world. In recent years, there have been significant developments in the theory of probability distribution, resulting in the creation of several general families of distributions. These distributions have been successfully applied to solve a wide range of statistical problems. For a complete grasp of this subject, see De Brito *et al.* (2019).

In many practical situations, classical distributions may not accurately represent real-world data (Khosa *et al.,* 2020). For example, when dealing with data that exhibit a monotonic hazard rate function (HRF), it is common to use the Rayleigh, exponential, or Weibull distributions. Among these models, the Weibull distribution is most commonly used to model natural phenomena (Yousof *et al.,* 2023). However, the Weibull model is not suitable for data that show a nonmonotonic HRF, such as unimodal, modified unimodal, or bathtub-shaped data (Ahmad *et al.,* 2019). To address these challenges, researchers have shown increasing interest in developing more flexible distributions. This is achieved by extending the classical distributions and introducing additional parameter(s) to the baseline model.

Over the last three decades, several families of distributions have been proposed and examined to model data in various applied fields, such as medical, social, environmental, engineering, and biological studies. Some well-known families include the exponential G family by Gupta & Kundu (1999), the transmuted G family by Shaw & Buckley (2007), the Marshall-Olkin G family by Marshall & Olkin (2007), the Kumaraswamy G family by Nadarajah *et al.* (2012)) and the Topp Leone-G family by Al-Shomrani *et al.* (2016), among many others.

Recently, the field of distribution theory has seen numerous contributions from researchers. In particular, trigonometric families of continuous distributions, such as the Sin-G family, have gained considerable attention in both statistical theory and practical applications (Jamal *et al.,* 2021). Mahmood *et al.,* 2019 introduced a new family of distributions called the sine-G family, which utilizes a trigonometric function. Another significant contribution to these families was made by Ampadu, 2021, who developed the hyperbolic Tan-X family. Additional families include the sine Kumaraswamy-G family Chesneau & Jamal, 2020, the sine Topp-Leone-G family Al-Babtain *et al.,* 2020, the alpha-sine-G family Benchiha *et al.,* 2023, the Sine Type II Topp-Leone-G family Isa *et al.,* 2023, the sine π power odd-g family Sapkota *et al.,* 2024, a new weighted sine-Weibull distribution Heydari *et al.,* 2024, and a novel sin-G class of distributions Ahmad *et al.,* 2024. These new families offer enhanced flexibility and applicability compared to existing models.

Maximum Likelihood Estimation (MLE) is widely recognized as the superior method for estimating parameters in various statistical models, including a family of probability distributions (see Adubisi *et al.,* 2024; Bandar *et al.,* 2023), due to its efficiency and consistency, particularly for large sample sizes Lehmann & Casella, 2006. Moreover, MLE is consistent, meaning that as the sample size increases, the estimators converge to the true parameters Cox, 2018. MLE is also robust in the presence of censored data, which is common in survival analysis, effectively handling right- and left-censoring, as evidenced by practical applications in survival and reliability models Bickel & Doksum, 2015. Therefore, in this study, we adapt MLE for parameter estimation.

Building on the previous discussion, we now present a novel approach to developing new probability models that eliminates the requirement for extra parameters. This innovative method is referred to as the New Transformed Sine G family distributions.

A New Transformed Sine G family is being established through trigonometric transformation and exponential truncation. This family will offer greater flexibility with a diverse range of distribution shapes, including skewed, bimodal, and non-monotonic behaviors. Furthermore, this approach is well-suited for modeling periodic or cyclical data, such as daily or yearly fluctuations, which traditional distributions may not effectively capture.

2. Proposed Distribution

A random variable X is said to have the New Transformed sine G family of distributions (NTS-G) if its cumulative density function (cdf) and probability density function (pdf) is expressed respectively as:

$$F(x;\varepsilon) = \frac{e^{\sin\left[\frac{\pi H(x)}{1+H(x)}\right]} - 1}{e-1} \tag{1}$$

$$f(x;\varepsilon) = \frac{\pi h(x) e^{\sin\left[\frac{\pi H(x)}{1+H(x)}\right]}}{[e-1]\left[1+H(x)\right]^2} \cos\left[\frac{\pi H(x)}{1+H(x)}\right] \tag{2}$$

Subtracting the cdf in equation 1 from 1, and simplyfying further, the survival function $S(x,\epsilon)$ of NTS-G is represented as;

$$S(x;\varepsilon) = \frac{e - e^{\sin\left[\frac{\pi H(x)}{1+H(x)}\right]}}{e-1} \tag{3}$$

The hazard function $H(x;\epsilon)$ of the NTS-G distribution was obtained by dividing the density function in 3 by the survivor function in 4 and simplifying further gives;

$$H(x;\varepsilon) = \frac{\pi h(x) e^{\sin\left[\frac{\pi H(x)}{1+H(x)}\right]}}{\left[1+H(x)\right]^2 \left[e - e^{\sin\left[\frac{\pi H(x)}{1+H(x)}\right]}\right]} \cos\left[\frac{\pi H(x)}{1+H(x)}\right] \tag{4}$$

The cumulative hazard function $H_r(x;\epsilon)$ of NTS-G is obtained by integrating the hazard function in equation 4 over time t and simplyfying further gives;

$$H_r(x;\varepsilon) = -\log\left(\frac{e - e^{\sin\left[\frac{\pi H(x)}{1+H(x)}\right]}}{e-1}\right) \tag{5}$$

2.1 Mathematical Properties of The Proposed Model

Here, we introduce some mathematical properties of the NTS-G family of distributions, which include the quantile function, the moment, and the moment-generating function.

2.1.1 Quantile Function

The quantile function of NTS-G family of distribution can be obtained by finding the inverse of the cummulative function in equation 1 which is expressed as;

$$x_u = H^{-1}\left(\frac{\arcsin\left[\log\left(u(1-e)\right)+1\right]}{\pi - \arcsin\left[\log\left(u(1-e)+1\right)\right]}\right) \quad ; \quad 0 < u < 1 \tag{6}$$

The median of the NTS-G family is obtained by substituting, $u = 0.5$ in equation 6which gives;

$$x_{0.5} = H^{-1}\left(\frac{\arcsin\left[\log\left(0.5(1-e)\right)+1\right]}{\pi - \arcsin\left[\log\left(0.5(1-e)+1\right)\right]}\right) \tag{7}$$

The first and third quartiles of NTS-G family can also be obtained similarly by substituting $u = 0.25$ and $u = 0.75$ in equation 6 given respectively in 8 and 9 as;

$$x_{0.25} = H^{-1}\left(\frac{\arcsin\left[\log\left(0.25(1-e)\right)+1\right]}{\pi - \arcsin\left[\log\left(0.25(1-e)+1\right)\right]}\right) \tag{8}$$

$$x_{0.75} = H^{-1}\left(\frac{\arcsin\left[\log\left(0.75(1-e)\right)+1\right]}{\pi - \arcsin\left[\log\left(0.75(1-e)+1\right)\right]}\right) \tag{9}$$

2.1.2 Moments

The r^{th} moment of the NTS-G random variable X is defined as:

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x;\varepsilon)dx \tag{10}$$

Substituting the pdf of NTS-G in equation (2) into 10, we obtained the r^{th} moment of the NTS-G distribution as;

$$\mu_r' = \int_{-\infty}^{\infty} x^r \frac{\pi h(x;\varepsilon)\cos\left(\frac{\pi H(x;\varepsilon)}{1+H(x;\varepsilon)}\right)}{(e-1)\left(1+H(x;\varepsilon)\right)^2} e^{\sin\left(\frac{\pi H(x;\varepsilon)}{1+H(x;\varepsilon)}\right)} dx \tag{11}$$

using power series:

$$e^y = \sum_{i=1}^{\infty} \frac{y^i}{i!} \tag{12}$$

Using equation (12)

$$e^{\sin\left(\frac{\pi H(x;\varepsilon)}{1+H(x;\varepsilon)}\right)} = \sum_{i=1}^{\infty} \frac{\sin\left(\frac{\pi H(x;\varepsilon)}{1+H(x;\varepsilon)}\right)}{i!} \tag{13}$$

Also we have that;

$$\frac{1}{(1+x)^2} = \sum_{j=1}^{\infty} (-1)^j j x^{j-1} \tag{14}$$

Using equation (14)

$$\frac{1}{(1+H(x;\varepsilon))^2} = \sum_{j=1}^{\infty} (-1)^j j\left[1+H(x;\varepsilon)\right]^{j-1} \tag{15}$$

Applying binomial expansion to equation (**15**)we have;

$$\frac{1}{(1+H(x;\varepsilon))} = \sum_{j=1}^{\infty}\sum_{k=0}^{j} j(-1)^j \binom{j-1}{k} \left[H(x;\varepsilon)\right] \tag{16}$$

Substituting equation (13) and (15) in (11) the r^{th} moment of the NTS-G distribution can be simplified as;

$$\mu_r' = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{j} \frac{j(-1)^j \binom{j-1}{k}}{i!(e-1)} \int_{-\infty}^{\infty} x^r h(x;\varepsilon)\cos\left(\frac{\pi H(x;\varepsilon)}{1+H(x;\varepsilon)}\right)\left[H(x;\varepsilon)\right]dx \tag{17}$$

2.1.3 Moment Generating Function

The moment generating function of NTS-G family of distribution can be derived as;

$$M_X(t) = E\left[e^{tx}\right] = \int_{-\infty}^{\infty} e^{tx} f(x;\varepsilon)dx \tag{18}$$

Substituting the pdf of NTS-G in equation 2 into 18, we obtained the r^{th} moment of the NTS-G distribution as;

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{\pi h(x;\varepsilon) e^{\sin\left(\frac{\pi H(x;\varepsilon)}{1+H(x;\varepsilon)}\right)}}{(e-1)\left[1+H(x)\right]^2} \cos\left(\frac{\pi H(x;\varepsilon)}{1+H(x;\varepsilon)}\right) dx \tag{19}$$

Applying (12) we have;

$$e^{tx} = \sum_{m=1}^{\infty} \frac{(tx)^i}{i!} = \sum_{m=1}^{\infty} \frac{t^i x^i}{i!} \tag{20}$$

so that;

$$e^{tx+\sin\left(\frac{\pi H(x;\varepsilon)}{1+H(x;\varepsilon)}\right)} = \sum_{i=1}^{\infty} \frac{\sin\left(\frac{\pi H(x;\varepsilon)}{1+H(x;\varepsilon)}\right)^i}{i!} \tag{21}$$

Applying the binomial expansion in equation 15 and 21 to equation (**18**)we have;

$$M_X(t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{j} \frac{j(-1)^j \binom{j-1}{k}}{i!(e-1)} \int_{-\infty}^{\infty} \pi h(x;\varepsilon)\left[H(x;\varepsilon)\right]\left[tx + \sin\left(\frac{\pi H(x;\varepsilon)}{1+H(x;\varepsilon)}\right)\right]^i \cos\left(\frac{\pi H(x;\varepsilon)}{1+H(x;\varepsilon)}\right) dx \tag{22}$$

2.2 Sub Models

Here we consider some of the special cases of the NTS-G family of distributions.

2.2.1 New Transformed Sine Weibull Distribution

Consider the Weibull distribution with a pdf and cdf given respectively as;

$$f(x,\lambda,\beta) = \frac{\lambda}{\beta} x^{\lambda-1} e^{-\left(\frac{x}{\beta}\right)^\lambda} \tag{23}$$

$$F(x,\lambda,\beta) = 1 - e^{\left(\frac{x}{\beta}\right)^\lambda} \tag{24}$$

Substituting equation (23) and (24) in (2) and (1) respectively, the pdf and cdf of the new transformed sine Weibull distribution can be obtained as;

$$f(x;\varepsilon) = \frac{\pi\frac{\lambda}{\beta} x^{\lambda-1} e^{-\left(\frac{x}{\beta}\right)^\lambda} e^{\sin\left[\frac{\pi-\pi e^{\left(\frac{x}{\beta}\right)^\lambda}}{2-e^{\left(\frac{x}{\beta}\right)^\lambda}}\right]}}{[e-1]\left[2-e^{\left(\frac{x}{\beta}\right)^\lambda}\right]^2} \cos\left[\frac{2\pi - \pi e^{\left(\frac{x}{\beta}\right)^\lambda}}{2-e^{\left(\frac{x}{\beta}\right)^\lambda}}\right] \tag{25}$$

$$F(x;\varepsilon) = \frac{e^{\sin\left[\frac{\pi-\pi e^{\left(\frac{x}{\beta}\right)^\lambda}}{2-e^{\left(\frac{x}{\beta}\right)^\lambda}}\right]} - 1}{e-1} \tag{26}$$

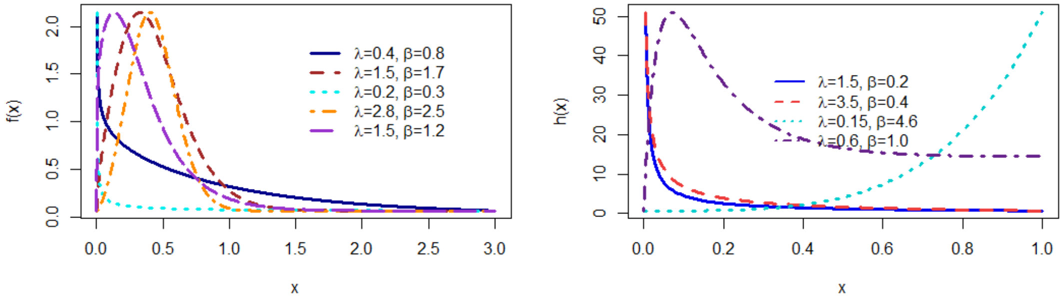

Figure 1. Probability density funcrion (PDF) and hazard function (h(x)) of NTSW distribution.

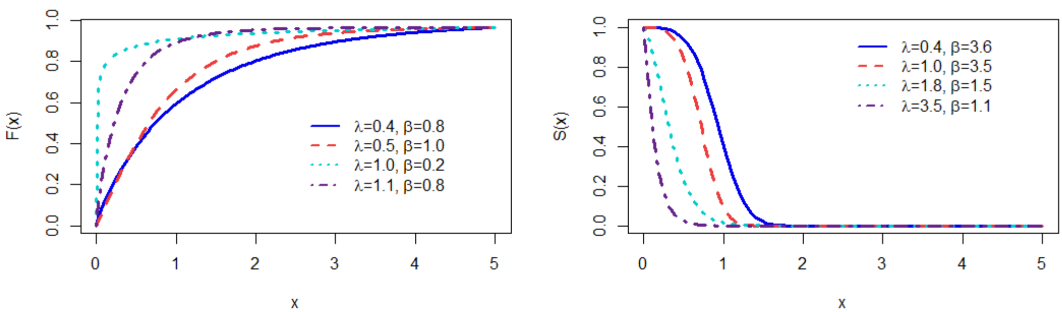

Figure 2. Cumulative density function (CDF) and survival function (S(x))of NTSW.

Figures 1 and 2 show plots for the pdf, hazard function, cdf, and survival function of the NTSW distribution. The pdf plot indicates that the NTSW distribution has a skewed and reversed-J shape, making it a suitable model for diverse datasets. Furthermore, the hazard function exhibits increasing, decreasing, and reverse bathtub shapes, suggesting that the NTSW distribution may be well-suited for modeling various types of lifetime data.

2.2.2 New Transformed Sine Frechet Distribution

Consider the Frechet distribution with pdf and cdf given respectively as;

$$f(x, \lambda, \beta) = \lambda\beta x^{-(\beta+1)} e^{-\lambda x^{-\beta}} \tag{27}$$

$$F(x, \lambda, \beta) = e^{-\lambda x^{-\beta}} \tag{28}$$

Substituting 27 and 28 in 2 and 1 respectively, the pdf and cdf of the New Transformed Sine Frechet distribution can be obtained as;

$$f(x; \varepsilon) = \frac{\lambda\beta\pi x^{-(\beta+1)} e^{\sin\left[\frac{\pi e^{-\lambda x^{-\beta}}}{1+e^{-\lambda x^{-\beta}}}\right] - \lambda x^{-\beta}}}{[e-1]\left[1 + e^{-\lambda x^{-\beta}}\right]^2} \cos\left[\frac{\pi e^{-\lambda x^{-\beta}}}{1 + e^{-\lambda x^{-\beta}}}\right] \tag{29}$$

$$F(x; \varepsilon) = \frac{e^{\sin\left[\frac{\pi e^{-\lambda x^{-\beta}}}{1+e^{-\lambda x^{-\beta}}}\right]} - 1}{e-1} \tag{30}$$

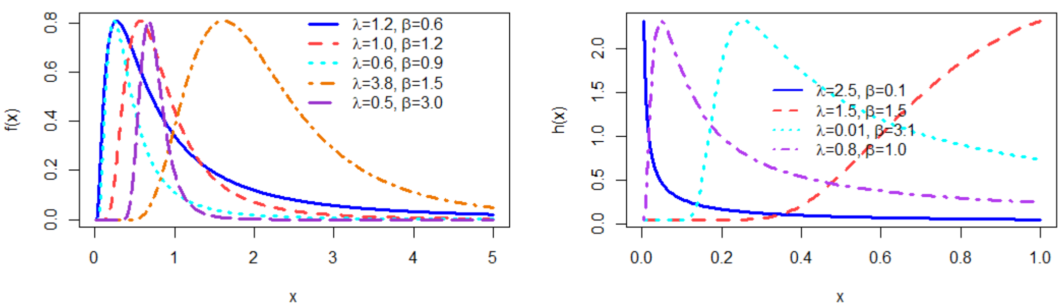

Figure 3. Probability density function (PDF) and Hazard function (h(x)) of NTSF distribution at a different parameter values.

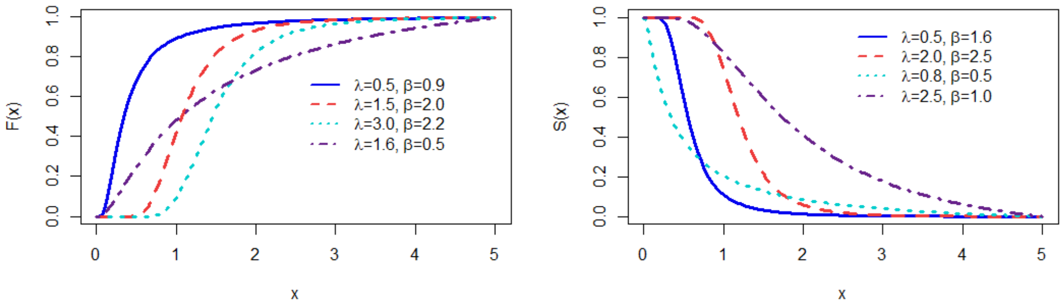

Figure 4. Cumulative density function (CDF) and Survival function (S(x)) of NTSF distribution at a different parameter values.

Figures 3 and 4 display the plots for pdf, hazard function, CDF, and survival function of the NTSF distribution. The pdf plot shows that the NTSF is skewed with different shapes, making it a suitable model for diverse datasets. In addition, the hazard function exhibits increasing, decreasing, and unimodal shapes, indicating that the NTSF may be suitable for modeling various types of lifetime data.

2.2.3 New Transformed Sine Kumarswamy Distribution

Consider the Kumarswamy distribution with pdf and cdf given, respectively, as

$$f(x, \lambda, \beta) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1} \tag{31}$$

$$F(x, \lambda, \beta) = 1-(1-x^{\alpha})^{\beta} \tag{32}$$

Substituting 31 and 32 in 2 and 1 respectively, the pdf and cdf of the New Transformed Sine Kumarswamy distribution can be obtained as;

$$f(x;\varepsilon) = \frac{\alpha\beta\pi x^{\alpha-1}(1-x^{\alpha})^{\beta-1} e^{\sin\left[\frac{\pi\left(1-(1-x^{\alpha})^{\beta}\right)}{2-(1-x^{\alpha})^{\beta}}\right]}}{[e-1]\left[2-(1-x^{\alpha})^{\beta}\right]^{2}} \cos\left[\frac{\pi\left(1-(1-x^{\alpha})^{\beta}\right)}{2-(1-x^{\alpha})^{\beta}}\right] \tag{33}$$

$$F(x;\varepsilon) = \frac{e^{\sin\left[\frac{\pi\left(1-(1-x^{\alpha})^{\beta}\right)}{2-(1-x^{\alpha})^{\beta}}\right]}-1}{e-1} \tag{34}$$

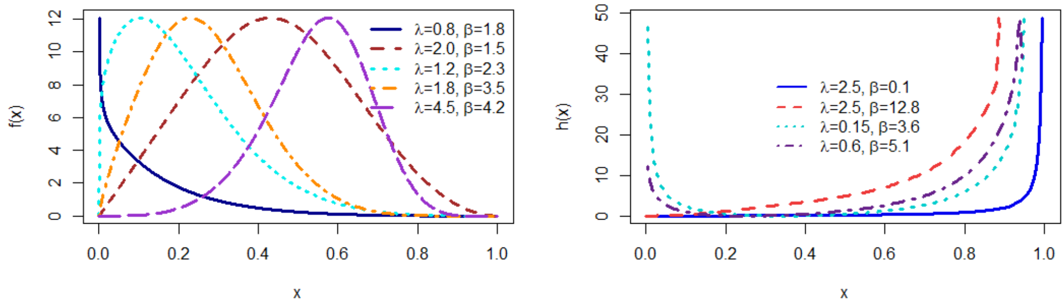

Figure 5. Probability density function (PDF) and Hazard function (h(x)) of NTSK distribution at a different parameter values.

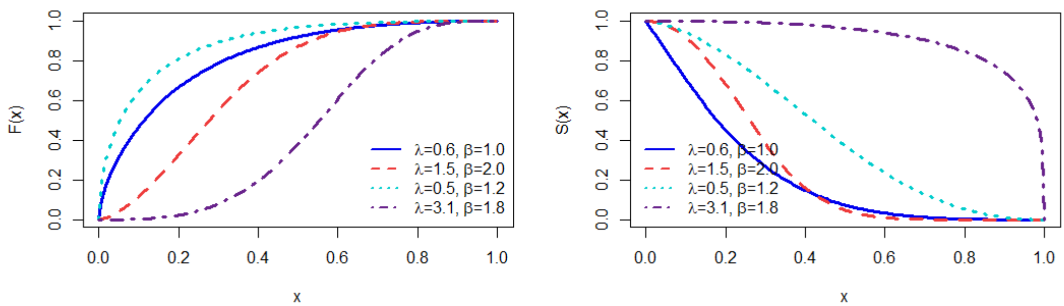

Figure 6. Cumulative density function (CDF) and Survival function (S(x)) of NTSK distribution at a different parameter values.

The plots for the pdf, hazard function, cdf, and survival function of the NTSK distribution are displayed in Fig.5 and Fig.6. The plot for the pdf reveals that the NTSK is skewed with various shapes, making it a good model for different kinds of dataset. Additionally, it can be observed that the hazard function exhibits increasing and bathtub shapes. This suggests that the NTSK may be suitable for modeling various lifetime data.

3. Estimation

Here we present an inferential study of NTS-G. We use the maximum likelihood estimation (MLE) method to obtain the mathematical expression of the parameter and its maximum likelihood estimates (MLEs) (for more information on MLE estimation, see Casella & Berger, 2024). Furthermore, a Monte Carlo simulation study is conducted to assess the consistency of the estimates.

Let $X_1, X_2, \ldots, X_n$ represent a random sample of size n taken from the NTS-G family of distributions. The likelihood function (ℓ) can be expressed as follows:

$$\ell(\varepsilon/X_1, X_2, ..., X_n) = \prod_{i=1}^{n} f(x_i, \varepsilon) \tag{35}$$

Substitute the PDF in 2

$$\ell(\varepsilon/X_1, X_2, ..., X_n) = \prod_{i=1}^{n} \frac{\pi h(x;\varepsilon) e^{\sin\left[\frac{\pi H(x;\varepsilon)}{1+H(x;\varepsilon)}\right]}}{(e-1)\left[1+H(x;\varepsilon)\right]} \cos\left[\frac{\pi H(x;\varepsilon)}{1+H(x;\varepsilon)}\right] \tag{36}$$

The log likelihood can be derive by taking the natural logarithm of the likelihood function in equation 36 as;

$$\log \ell(\varepsilon/x_i) = n \log \pi + \sum_{i=1}^{n} \log h(x;\varepsilon) + \sum_{i=1}^{n} \sin A + \sum_{i=1}^{n} \log \cos A - n \log(e-1) - 2\sum_{i=1}^{n} \log\left[1+H(x;\varepsilon)\right] \tag{37}$$

Where; $A = \left[\frac{\pi H(x;\varepsilon)}{1+H(x;\varepsilon)}\right]$

To obtain the estimates of the parameter vector ε, we partially differentiate equation 38 with respect to ε and set the result equal to zero. This simplifies to:

$$\frac{\partial \ell(\varepsilon/x_i)}{\partial \varepsilon} = \sum_{i=1}^{n} \frac{h'(x)}{h(x)} + \sum_{i=1}^{n} \cos A \frac{\pi h(x;\varepsilon)}{\left[1+H(x;\varepsilon)\right]^2} - \sum_{i=1}^{n} \frac{\sin A \pi h(x;\varepsilon)}{\cos A\left[1+H(x;\varepsilon)\right]^2} - 2\sum_{i=1}^{n} \frac{h(x;\varepsilon)}{1+H(x;\varepsilon)} \tag{38}$$

Equation 38 is nonlinear; therefore, it cannot be solved analytically. Hence, we need to employ numerical methods for the solution.

4. Simulation

In this section, we present a simulation study to evaluate the performance of maximum likelihood estimates for the parameters. The study involves generating random samples of various sizes from the NTSW distribution. The sample sizes considered are 50, 100, 250, 500, and 1000. For each sample, we calculate the maximum likelihood estimates of the parameters β and λ. These steps are repeated 1000 times, and then we compute the average estimates, average bias, and root mean square error.

The results of the simulation study are shown in Tables 1 and 2 below. Based on the predefined values, we observe that the estimated values of the parameters are close to the predefined values. Additionally, we note that the bias and mean square error of the estimates decrease as the sample size increases. These results indicate that the maximum likelihood estimates of the parameters are consistent.

Table 1. The outcome of simulation for the model's parameter estimation based on MLE

SET one							
			β			λ	
Actual values	Sample size	Means	Biases	RMSE	Means	Biases	RMSE
	50	1.5442	0.0442	0.1931	1.2703	0.0703	0.2349
$\beta = 1.5$	100	1.5232	0.0232	0.1311	1.2305	0.0305	0.1435
$\lambda = 1.2$	250	1.5091	0.0091	0.0795	1.2096	0.0096	0.0863
	500	1.5024	0.0024	0.0526	1.2036	0.0036	0.0593
	1000	1.5004	0.0004	0.0384	1.2029	0.0029	0.0408

Table 2. The outcome of simulation for the model's parameter estimation based on MLE

SET two							
			λ			β	
Actual values	Sample size	Means	Biases	RMSE	Means	Biases	RMSE
	50	0.5148	0.0148	0.0644	1.0509	0.0509	0.1791
$\beta = 0.5$	100	0.5078	0.0078	0.0437	1.0218	0.0218	0.1105
$\lambda = 1.0$	250	0.5030	0.0030	0.0265	1.0065	0.0065	0.0670
	500	0.5008	0.0008	0.0175	1.0025	0.0025	0.0464
	1000	0.5001	0.0001	0.0128	1.0023	0.0023	0.0319

5. Application

In this section, we apply the NTSW model to two real-life datasets and evaluate and compare the performance of four selected models: the four-parameter Weibull Exponentiated Weibull (WEW) model, the four-parameter Generalized Exponentiated Weibull (GEM) model, the four-parameter Kumarswammy Weibull (KWWEI) model, and the three-parameter Weibull Weibull (WW) model. We assess and compare the performance of these models using several model selection criteria, including the AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), and BIC (Bayesian Information Criterion). Additionally, we employ goodness-of-fit criteria such as the Anderson-Darling (AD) and KS statistics, along with the corresponding KS p-values. The general guideline is that smaller values of the AD and KS statistics, AIC, CAIC, and BIC, as well as larger values of the KS p-values, indicate a better fit of the corresponding model to the data.

Dataset I: This dataset consists of 100 observations measuring the amount of time clients wait before receiving the desired service at a bank. The data was used in ZeinEldin *et al.,* 2021. The dataset is "0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 2.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23, 27, 31.6, 33.1, 38.5"

Dataset II: The second dataset consists of the lifetimes of 50 industrial devices that were subjected to a life test at time zero, as reported by Aarset, 1987. The dataset is "0.1, 0.2, 1.0, 1.0, 1.0, 1.0, 1.0, 2.0, 3.0, 6.0,7.0, 11.0, 12.0, 18.0, 18.0,18.0, 18.0, 18.0, 21.0, 32.0,36.0, 40.0, 45.0, 45.0, 47.0, 50.0, 55.0, 60.0,63.0,63.0, 67.0, 67.0, 67.0, 67.0, 72.0, 75.0, 79.0, 82.0, 82.0,3.0, 84.0, 84.0, 84.0, 85.0, 85.0, 85.0, 85.0, 85.0, 86.0,86.0"

Table 3. The model parameters MLE estimates and information criteria for the dataset one

Model	α	θ	λ	β	AIC	CAIC	BIC
WEW	98.07	9.850	0.0709	79.07	**644.33**	**644.75.12**	**654.75**
GEW	21.89	0.2842	1.5640	80.42	653.90	654.33	664.33
KWWEI	3.9069	94.36	0.4450	96.45	642.60	643.03	653.02
WW	100.00	-	0.0104	0.6762	642.65	642.90	650.47
NTSW	-	-	0.0174	1.3710	639.97	640.09	645.18

Table 4. The test results for the Goodness-of-fit for dataset one

Model	A^*	W^*	KS	KS p-value
WEW	0.4080	0.0653	0.0561	0.9080
GEW	0.9944	0.1409	0.0910	0.3784
KWWEI	0.3862	0.0625	0.0527	0.9443
WW	0.4421	0.0712	0.0610	0.8511
NTSW	0.2857	0.0456	0.0485	0.9727

Tables 3 and 4 show that the NTSW model has the lowest values for AIC, BIC, CAIC, A^*, W^*, and K-S, with the highest P-value when compared to other alternative distributions. Based on these results, it is concluded that the NTSW model is the most suitable for fitting the first dataset among all the competing distributions. In summary, the NTSW distribution outperforms all other alternatives and is recommended as a valuable alternative for statistical research.

Furthermore, Figure 7 shows the histogram and estimated densities, along with the empirical cdf (ecdf) and the estimated cdf of the fitted model for the first dataset. This clearly demonstrates that the NTSW model is an appropriate and efficient model for the fitted pdf and cdf.

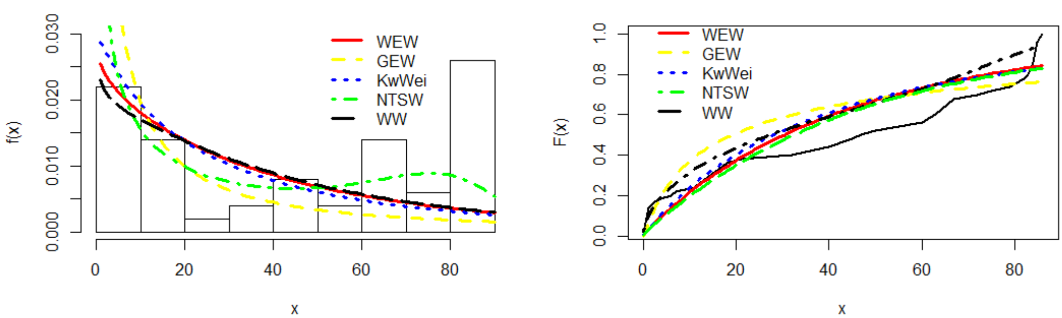

Figure 7. Application for data set 1.

Based on the findings from Tables 5 and 6, it is clear that the NTSW model exhibits the lowest values for the selection criteria measures. Moreover, it demonstrates the highest P-value of KS

Table 5. The model parameters MLE estimates and information criteria for the dataset two

Model	α	θ	λ	β	AIC	CAIC	BIC
WEW	100.00	9.8589	0.0464	78.8770	490.33	491.22	497.98
GEW	24.0748	0.1539	0.3893	76.9698	521.9063	522.80	529.55
KWWEI	8.5240	102.7848	0.1555	106.1077	496.68	496.68	503.44
WW	4.3735	-	0.1419	1.1897	492.19	492.71	497.93
NTSW	-	-	0.0139	1.8981	486.90	487.16	490.73

Table 6. The test results for the Goodness-of-fit for dataset two

Model	A^*	W^*	KS	KS p-value
WEW	3.0244	0.4994	0.1943	0.0458
GEW	4.8405	0.8766	0.2357	0.0078
KWWEI	0.3519	0.5640	0.2073	0.0272
WW	3.5490	3.2737	0.1923	0.0495
NTSW	3.0059	0.4961	0.1775	0.0858

when compared to all other competing distributions. Therefore, we can confidently conclude that the NTSW distribution is the most appropriate choice for fitting the second dataset, surpassing all other competing distributions. Furthermore, as illustrated in Fig.8, the NTSW model accurately represents both the fitted pdf and cdf, further solidifying its status as the optimal model for this dataset.

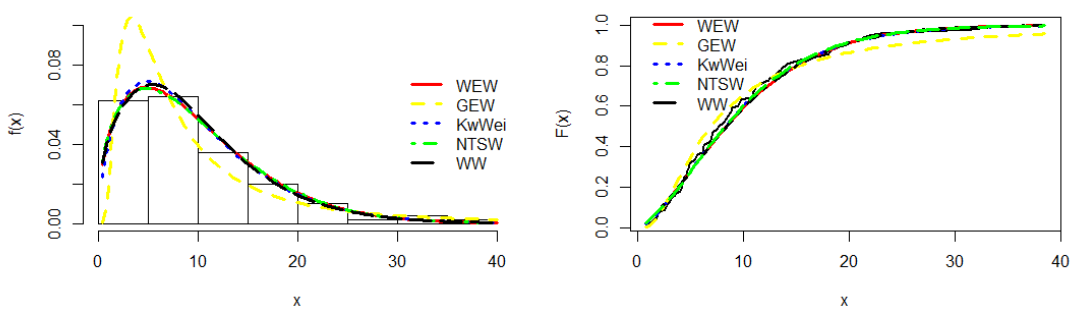

Figure 8. Application for data set 2.

6. Conclusions

This article introduces the New Transformed Sine-G (NTS-G) family, a new family of probability distributions that extends the Sin-G family of continuous distributions. It derives the mathematical properties of the model, including hazard and survival functions, moments, and the quantile function. Model parameter estimation is conducted using the maximum likelihood method, and a Monte Carlo simulation study evaluates the performance of this estimation, indicating that the maximum likelihood estimates (MLEs) are consistent. Additionally, the article develops specific sub-models based on the Weibull, Fréchet, and Kumaraswamy distributions. The application section focuses on the new Transformed Sin Weibull (NTSW) distribution, highlighting its potential for analyzing and modeling two real-life data sets. The NTSW model demonstrates a superior ability to fit these two lifetime data sets compared to four competing models, some of which have more parameters.

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Conflicts of Interest

The authors declare no conflict of interest.

Author Contributions

Conceptualization: OSI, A. A.; RAGHAV, Y. S.; SABO, S. A.; MUSA, I. Z. **Data curation**: OSI, A. A.; RAGHAV, Y. S.; SABO, S. A.; MUSA, I. Z. **Formal analysis**: OSI, A. A.; RAGHAV, Y. S.; SABO, S. A.; MUSA, I. Z. **Funding acquisition**: OSI, A. A.; RAGHAV, Y. S.; SABO, S. A.; MUSA, I. Z. **Investigation**: OSI, A. A.; RAGHAV, Y. S.; SABO, S. A.; MUSA, I. Z. **Methodology**: OSI, A. A.; RAGHAV, Y. S.; SABO, S. A.; MUSA, I. Z. **Project administration**: OSI, A. A.; RAGHAV, Y. S.; SABO, S. A.; MUSA, I. Z. **Software**: OSI, A. A.; RAGHAV, Y. S.; SABO, S. A.; MUSA, I. Z. **Resources**: OSI, A. A.; RAGHAV, Y. S.; SABO, S. A.; MUSA, I. Z. **Supervision**: OSI, A. A.; RAGHAV, Y. S.; SABO, S. A.; MUSA, I. Z. **Validation**: OSI, A. A.; RAGHAV, Y. S.; SABO, S. A.; MUSA, I. Z. **Visualization**: OSI, A. A.; RAGHAV, Y. S.; SABO, S. A.; MUSA, I. Z. **Writing – original draft**: OSI, A. A.; RAGHAV, Y. S.; SABO, S. A.; MUSA, I. Z. **Writing – review and editing:** OSI, A. A.; RAGHAV, Y. S.; SABO, S. A.; MUSA, I. Z.

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